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Gröbner–Shirshov Bases for Lie Superalgebras and Their Universal Enveloping Algebras

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We show that a set of monic polynomials in a free Lie superalgebra is a Gröbner–Shirshov basis for a Lie superalgebra if and only if it is a Gröbner–Shirshov basis for its universal enveloping algebra. We investigate the structure of Gröbner–Shirshov bases for Kac–Moody superalgebras and give explicit constructions of Gröbner–Shirshov bases for classical Lie superalgebras. © 1999 Academic Press

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1. INTRODUCTION

Let $\mathscr{A}$ be a free (commutative, associative, or Lie) algebra over a field k, let $S \subset \mathscr{A}$ be a set of relations in $\mathscr{A}$, and let $\langle S \rangle$ be the ideal of $\mathscr{A}$ generated by S. One of the fundamental problems in the theory of abstract algebras is the *reduction problem*: given an element $f \in \mathscr{A}$, one would like to find a *reduced expression* for f with respect to the relations in S. One of the most common approaches to this problem is to find another set of generators for the relations in S that can replace the original relations so that one can get an effective algorithm for the reduction problem. More precisely, if one can find a set S^c of generators of the ideal $\langle S \rangle$ which is *closed* under a certain *composition* of relations in S, then there exists an easy criterion by which one can determine whether an element $f \in \mathscr{A}$ is reduced with respect to S or not.

In 1965, inspired by Gröbner's suggestion, Buchberger found a criterion and an algorithm of computing such a set of generators of the ideals for commutative algebras [16], which were modified and refined in [17, 18]. Such a set of generators of ideals is now referred to as a *Gröbner basis*, and it has become one of the most popular research topics in the theory of commutative algebras (see, for example, [3]). In 1978, Bergman developed the theory of Gröbner bases for associative algebras by proving the *diamond lemma* [4]. His idea is a generalization of Buchberger's theory and it has many applications to various areas of the theory of associative algebras such as quantum groups.

For the case of Lie algebras, where the situation is more complicated than commutative or associative algebras, the parallel theory of Gröbner basis was developed by Shirshov in 1962 [30], which is even earlier than Buchberger's discovery. In that paper, which was written in Russian and never translated in English, he introduced the notion of *composition* of elements of a free Lie algebra and showed that a set of relations which is closed under the composition has the desired property. Shirshov's idea is essentially the same as that of Buchberger, and it was noticed by Bokut that Shirshov's method works for associative algebras as well [7]. For this reason, we will call such a set of relations of a free Lie algebra (and of a free associative algebra) a *Gröbner–Shirshov basis*. (See [2] for a more detailed history of Gröbner–Shirshov basis.) It has been used to determine the solvability of some word problems [29, 30, 6] and to prove some embedding theorems [5, 7, 8]. In a series of works by Bokut, Klein, and Malcolmson, Gröbner–Shirshov bases for finite-dimensional simple Lie algebras and for the quantized enveloping algebra of type A_n were constructed explicitly ([9–11, 14]).

In this work, we develop the theory of Gröbner–Shirshov bases for Lie superalgebras and their universal enveloping algebras. This paper is organized as follows. In Section 2, after introducing the basic facts such as *super-Lyndon–Shirshov words* (*monomials*) and *composition lemma*, we prove that a set of monic polynomials in a free Lie superalgebra is a Gröbner–Shirshov basis for a Lie superalgebra if and only if it is a Gröbner–Shirshov basis for its universal enveloping algebra (Theorem 2.8). This is a generalization of the corresponding result for Lie algebras obtained in [15]. Thus the theory of Gröbner–Shirshov bases for Lie superalgebras and that of associative algebras are unified in this way, and as a by-product, we obtain a purely combinatorial proof of the Poincaré–Birkhoff–Witt theorem (Proposition 2.11).

In Section 3, we investigate the structure of Gröbner–Shirshov bases for Kac–Moody superalgebras and prove that, in order to find a Gröbner–Shirshov basis for a Kac–Moody superalgebra, it suffices to consider the completion of Serre relations of the positive part (or negative part) which is closed under the composition (Theorem 3.5). As a corollary, we obtain the *triangular decomposition* of Kac–Moody superalgebras and their universal enveloping algebras (Corollary 3.6). Our result in this section is a generalization of the corresponding result for Kac–Moody algebras obtained in [14].

Finally, in Section 4, we give an explicit construction of Gröbner–Shirshov bases for classical Lie superalgebras. The outline of our construction can be described as follows. We first start with a Kac–Moody superalgebra which is isomorphic to a given classical Lie superalgebra. Using the supersymmetry and Jacobi identity, we expand the set of Serre relations to a set R of relations and determine the set B of R-reduced super-Lyndon–Shirshov monomials. Now comparing the number of elements of B with the dimension of the corresponding classical Lie superalgebra, we conclude that the set R is indeed a Gröbner–Shirshov basis.

2. GRÖBNER–SHIRSHOV BASES FOR LIE SUPERALGEBRAS

Let $X = X_{\bar{0}} \cup X_{\bar{1}}$ be a $\mathbb{Z}_2$-graded set with a linear ordering $\prec$, and let X^* (resp., $X^\#$) be the semigroup of associative words on X (resp., the groupoid of nonassociative words on X). Then the semigroup X^* (resp., the groupoid $X^\#$) has the $\mathbb{Z}_2$-grading $X^* = X^*_{\bar{0}} \oplus X^*_{\bar{1}}$ (resp., $X^\# = X^\#_{\bar{0}} \oplus X^\#_{\bar{1}}$) induced by that of X. The elements of $X^*_{\bar{0}}$ and $X^\#_{\bar{0}}$ (resp., $X^*_{\bar{1}}$ and $X^\#_{\bar{1}}$) are called *even* (resp., *odd*).

We denote by $l(u)$ the *length* of a word u and the empty word is denoted by 1. For an associative word $u \in X^*$, we can choose a certain arrangement of brackets on u, which will be denoted by (u). Conversely, there is a canonical bracket removing homomorphism ρ: $X^\# \to X^*$ given by $\rho((u)) = u$ for $u \in X^*$.

We consider two linear orderings $<$ and $\ll$ on X^* defined as follows:

(i) $u < 1$ for any nonempty word u; and inductively, $u < v$ whenever $u = x_i u'$, $v = x_j v'$, and $x_i \prec x_j$ or $x_i = x_j$ and $u' < v'$.

(ii) $u \ll v$ if $l(u) < l(v)$ or $l(u) = l(v)$ and $u < v$.

The ordering $<$ (resp., $\ll$) is called the *lexicographical ordering* (resp., *length-lexicographical ordering*). We define the orderings $<$ and $\ll$ on $X^\#$ by (i) $u < v$ if and only if $\rho(u) < \rho(v)$, and (ii) $u \ll v$ if and only if $\rho(u) \ll \rho(v)$.

A nonempty word u is called a *Lyndon–Shirshov word* if $u \in X$ or $vw > wv$ for any decomposition of $u = vw$ with $v, w \in X^*$. A nonempty word u is called a *super-Lyndon–Shirshov word* if either it is a Lyndon–Shirshov word or it has the form $u = vv$ with v a Lyndon–Shirshov word in X_1^*. A nonempty nonassociative word u is called a *Lyndon–Shirshov monomial* if either u is an element of X or

(i) if $u = u_1 u_2$, then u_1, u_2 are Lyndon–Shirshov monomials with $u_1 > u_2$,

(ii) if $u = (v_1 v_2) w$ then $v_2 \le w$.

A nonempty nonassociative word u is called a *super-Lyndon–Shirshov monomial* if either it is a Lyndon–Shirshov monomial or it has the form $u = vv$ with v a Lyndon–Shirshov monomial in $X_1^\#$.

Remark. In some literatures, the Lyndon–Shirshov words have been referred to as *regular words*, *normal words*, *Lyndon words*, etc. Since the definition of Lyndon–Shirshov words dates back to the works by Chen, Fox, and Lyndon [19] and Shirshov [27], we decide to call them Lyndon–Shirshov words. The definition of super-Lyndon–Shirshov words can be found in [1, 24].

The following lemma asserts that there is a natural 1-1 correspondence between the set of super-Lyndon–Shirshov words and the set of super-Lyndon–Shirshov monomials.

LEMMA 2.1 ([1, 19, 24, 26]). *If u is a super-Lyndon–Shirshov monomial, then $\rho(u)$ is a super-Lyndon–Shirshov word. Conversely, for any super-Lyndon–Shirshov word u, there is a unique arrangement of brackets $[u]$ on u such that $[u]$ is a super-Lyndon–Shirshov monomial.*

Let k be a field with $\mathrm{char}(k) \neq 2, 3$, and let $\mathscr{A}_X$ be the free associative algebra generated by X over k. The algebra $\mathscr{A}_X$ becomes a Lie superalgebra with the superbracket defined by

$$[x, y] = xy - (-1)^{(\deg x)(\deg y)} yx$$

for $x, y \in \mathscr{A}_X$. Let $\mathscr{L}_X$ be the subalgebra of $\mathscr{A}_X$ generated by X as a Lie superalgebra. Then $\mathscr{L}_X$ is the free Lie superalgebra generated by X over k. As we can see in the following theorem, there is a canonical linear basis for the free Lie superalgebra $\mathscr{L}_X$:

THEOREM 2.2 ([1, 19, 24, 26]). *The set of super-Lyndon–Shirshov monomials form a linear basis of the free Lie superalgebra $\mathscr{L}_X$ generated by X.*

Remark. The existence of linear bases for free Lie algebras of this form was first suggested by Hall [22], and later by Shirshov in a more general form ([26, 28]). The linear basis for a free Lie superalgebra given in the above theorem is called the *Lyndon–Shirshov basis*. It is a special case of the *Hall–Shirshov basis*.

Given a nonzero element $p \in \mathscr{A}_X$ we denote by $\bar{p}$ the maximal monomial appearing in p under the ordering $\ll$. Thus $p = \alpha\bar{p} + \sum \beta_i w_i$ with $\alpha, \beta_i \in k$, $w_i \in X^*$, $\alpha \neq 0$, and $w_i \ll \bar{p}$. The coefficient α of $\bar{p}$ is called the *leading coefficient* of p and p is said to be *monic* if $\alpha = 1$.

The following lemma plays a crucial role in defining the notion of *Lie composition.*

LEMMA 2.3 ([19, 24, 26]). *Let u and v be super-Lyndon–Shirshov words such that v is contained in u as a subword. Write $u = avb$ with $a, b \in X^*$. Then there is an arrangement of brackets $[u] = (a[v]b)$ on u such that $[v]$ is a super-Lyndon–Shirshov monomial, $\overline{[u]} = u$ and the leading coefficient of $[u]$ is either 1 or 2.*

Let $u = avb$ be a super-Lyndon–Shirshov word, where v is a super-Lyndon–Shirshov subword and $a, b \in X^*$. We define the *bracket on u relative to v*, denoted by $[u]_v$, as:

(i) $[u]_v = (a[v]b)$ if the leading coefficient of $[u]$ is 1,

(ii) $[u]_v = \frac{1}{2}(a[v]b)$ if the leading coefficient of $[u]$ is 2,

where the arrangement of brackets $[u]$ on u is the one described in Lemma 2.3. Note that $[u]_v$ is monic and $\overline{[u]_v} = u$.

Similarly, if p is a monic polynomial in the free Lie superalgebra $\mathscr{L}_X$ such that $\bar{p}$ is super-Lyndon–Shirshov, then we define the *bracket on u relative to p*, denoted by $[u]_p$ to be the result of the substitution of p instead of $\bar{p}$ in $[u]_{\bar{p}}$. Clearly, $[u]_p$ is monic and $\overline{[u]_p} = u$.

We now define the notion of *associative composition* of the elements in the free associative algebra $\mathscr{A}_X$ generated by X. Let p, q be monic elements in $\mathscr{A}_X$ with leading terms $\bar{p}$ and $\bar{q}$. If there exist $a, b \in X^*$ such that $\bar{p}a = b\bar{q} = w$ with $l(\bar{p}) > l(b)$, then we define the *composition of intersection* $(p, q)_w$ to be

$$(p, q)_w = pa - bq. \tag{2.1}$$

If there exist $a, b \in X^*$ such that $\bar{p} = a\bar{q}b = w$, then we define the *composition of inclusion* to be

$$(p, q)_w = p - aqb. \tag{2.2}$$

Note that we have $\overline{(p, q)_w} \ll w$ in either case.

Next we proceed to define the notion of *Lie composition* of the elements in the free Lie superalgebra $\mathscr{L}_X$ generated by X. Let p, q be monic polynomials in the free Lie superalgebra $\mathscr{L}_X$ with leading terms $\bar{p}$ and $\bar{q}$. If there exist $a, b \in X^*$ such that $\bar{p}a = b\bar{q} = w$ with $l(\bar{p}) > l(b)$, then we define the *composition of intersection* $\langle p, q\rangle_w$ to be

$$\langle p, q\rangle_w = [w]_p - [w]_q. \tag{2.3}$$

If there exist $a, b \in X^*$ such that $\bar{p} = a\bar{q}b = w$, then we define the *composition of inclusion* to be

$$\langle p, q\rangle_w = p - [w]_q. \tag{2.4}$$

We have $\overline{\langle p, q\rangle_w} \ll w$ in this case, too.

Remark. Our definition of Lie composition is essentially the same as the one given in [6, 23, 24, 29]. We modified the definition in [6, 23, 24, 29] to define the Lie composition $\langle p, q\rangle_w$ at one stroke.

Let S be a set of monic polynomials in $\mathscr{L}_X \subset \mathscr{A}_X$, let I be the (Lie) ideal generated by S in the free Lie superalgebra $\mathscr{L}_X$, and let J be the (associative) ideal generated by S in the free associative algebra $\mathscr{A}_X$. We denote by $L = \mathscr{L}_X / I$ the Lie superalgebra generated by X with defining relations S and let $\mathscr{U}(L) = \mathscr{A}_X / J$ be its universal enveloping algebra.

For $f, g \in \mathscr{A}_X$ and $w \in X^*$, we write $f \equiv_A g \mod(S, w)$ if $f - g = \sum \alpha_i a_i s_i b_i$, where $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$ with $a_i \bar{s}_i b_i \ll w$ for each i. Similarly, for $f, g \in \mathscr{L}_X$ and $w \in X^*$, we write $f \equiv_L g \mod(S, w)$ if $f - g = \sum \alpha_i (a_i (s_i) b_i)$, where $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$ with $\overline{(a_i(s_i)b_i)} \ll w$ for each i. The set S is said to be *closed under the associative composition* (resp., *Lie composition*) if for any $f, g \in S$, we have $(f, g)_w \equiv_A 0$ (resp., $\langle f, g\rangle_w \equiv_L 0$) $\mod(S, w)$.

A set of monic polynomials S in the free Lie superalgebra $\mathscr{L}_X$ is called a *Gröbner–Shirshov basis* for the ideal J (resp., for the ideal I) if it is closed under the associative composition (resp., Lie composition). By abuse of language, we also refer to S as a Gröbner–Shirshov basis for the associative algebra $\mathscr{U}(L)$ and for the Lie superalgebra L, respectively. An associative word u is said to be *S-reduced* if $u \neq a\bar{s}b$ for any $s \in S$ and $a, b \in X^*$. A nonassociative word u is said to be *S-reduced* if $\rho(u)$ is S-reduced.

The following lemma is a generalization of Lemma 1 in [9].

LEMMA 2.4. (a) *Every nonempty word u in the free associative algebra $\mathscr{A}_X$ can be written as*

$$u = \sum \alpha_i u_i + \sum \beta_j a_j s_j b_j, \tag{2.5}$$

where u_i is an S-reduced word, $\alpha_i, \beta_j \in k$, $a_j, b_j \in X^*$, $s_j \in S$, and $a_j \bar{s}_j b_j \underline{\ll} u$ for all i, j. Hence the set of S-reduced words spans the algebra $\mathscr{U}(L)$.

(b) *Every super-Lyndon–Shirshov monomial u in $\mathscr{L}_X$ can be written as*

$$u = \sum \alpha_i u_i + \sum \beta_j \big(a_j (s_j) b_j \big), \tag{2.6}$$

where u_i is an S-reduced super-Lyndon–Shirshov monomial, $\alpha_i, \beta_j \in k$, $a_j, b_j \in X^*$, $s_j \in S$, and $\overline{\big(a_j (s_j) b_j \big)} \underline{\ll} \bar{u}$ for all i, j. Hence the set of S-reduced super-Lyndon–Shirshov monomials spans the Lie superalgebra L.

Proof. Since the proof of (a) is similar to that of (b), we only give a proof of (b). If u is S-reduced, we are done. Thus we assume that $\bar{u} = a\bar{s}b$ for some $s \in S$, $a, b \in X^*$. Then $\bar{u}$ and $\bar{s}$ are super-Lyndon–Shirshov words and $\overline{u - \alpha[\bar{u}]_s} \ll \bar{u}$ for some $\alpha \in k$. Since $u - \alpha[\bar{u}]_s$ is a linear combination of super-Lyndon–Shirshov monomials whose leading terms are less than $\bar{u}$, we may proceed by induction, which completes the proof. ∎

The following lemma plays a crucial role in our discussion of Gröbner-Shirshov bases. It is originally due to Shirshov [30] and is now known as the *composition lemma*.

LEMMA 2.5 (cf. [1, 6, 24, 30]). *If S is a Gröbner–Shirshov basis for the ideal J, then for any $f \in J$, the word $\bar{f}$ contains a subword $\bar{s}$ with $s \in S$.*

It is clear that if a polynomial $f \in \mathscr{L}_X$ satisfies $f \equiv_L 0 \bmod(S, w)$ for $w \in X^*$, then $f \equiv_A 0 \bmod(S, w)$. The converse is also true if S is closed under the associative composition.

LEMMA 2.6. *Assume that S is closed under the associative composition. If a polynomial $f \in \mathscr{L}_X$ satisfies $f \equiv_A 0 \bmod(S, w)$ for $w \in X^*$, then $f \equiv_L 0 \bmod(S, w)$.*

Proof. Suppose $f \equiv_A 0 \bmod(S, w)$ for $w \in X^*$ and our assertion holds for all $w' \ll w$. Then $f \in J$, and by the composition lemma, $\bar{f} = a\bar{s}b$ for some $a, b \in X^*$ and $s \in S$. Since $f - [\bar{f}]_s \equiv_A 0 \bmod(S, \bar{f})$ and $\bar{f} \ll w$, our assertion follows by induction. ∎

LEMMA 2.7. *Let $f, g \in S$ be monic polynomials in $\mathscr{L}_X$ such that the associative composition $(f, g)_w$ is defined. Then we have*

$$(f, g)_w \equiv_A \langle f, g\rangle_w \bmod(S, w). \tag{2.7}$$

Proof. We consider the composition of intersection only. The proof for the composition of inclusion is similar. Recall that $[w]_f = fa + \sum \alpha_i a_i f b_i$ with $a_i \bar{f} b_i \ll w$ and $[w]_g = bg + \sum \beta_i c_i g d_i$ with $c_i \bar{g} d_i \ll w$. Thus $\langle f, g\rangle_w = [w]_f - [w]_g = fa - bg + h = (f, g)_w + h$, where $h \equiv_A 0 \bmod(S, w)$. Hence $(f, g)_w \equiv_A \langle f, g\rangle_w \bmod(S, w)$. ∎

Combining Lemmas 2.6 and 2.7, we obtain the main result of this section, which is a generalization of the main theorem in [15].

THEOREM 2.8. *Let S be a set of monic polynomials in the free Lie superalgebra $\mathscr{L}_X$. Then S is a Gröbner–Shirshov basis for the Lie superalgebra $L = \mathscr{L}_X/I$ if and only if S is a Gröbner–Shirshov basis for its universal enveloping algebra $\mathscr{U}(L) = \mathscr{A}_X/J$. That is, S is closed under the Lie composition if and only if it is closed under the associative composition.*

The following proposition, which is a generalization of Proposition 2 in [9], provides us with a criterion for determining whether a set of monic polynomials in the free Lie superalgebra is a Gröbner–Shirshov basis or not.

PROPOSITION 2.9. (a) *If the set of S-reduced words is a linear basis of $\mathscr{U}(L) = \mathscr{A}_X/J$, then S is a Gröbner–Shirshov basis for the ideal J of $\mathscr{A}_X$.*

(b) *If the set of S-reduced super-Lyndon–Shirshov monomials is a linear basis of $L = \mathscr{L}_X/I$, then S is a Gröbner–Shirshov basis for the ideal I of $\mathscr{L}_X$.*

Proof. Since the proof of (b) is the same as (a), we prove (a) only. Suppose on the contrary that S is not closed under the associative composition. Then there exist $f, g \in S$ such that $(f, g)_w \not\equiv_A 0 \bmod(S, w)$ for $w \in X^*$. By Lemma 2.4, we may write

$$(f, g)_w = \sum \alpha_i u_i + \sum \beta_j a_j s_j b_j,$$

where $\alpha_i, \beta_j \in k$, u_i is S-reduced, $a_j, b_j \in X^*$, $s_j \in S$, and $a_j \bar{s}_j b_j \ll w$ for all i and j. Since $(f, g)_w \not\equiv_A 0 \bmod(S, w)$, we have $\sum \alpha_i u_i \neq 0$ in $\mathscr{A}_X$. Since the set of S-reduced words is a linear basis of $\mathscr{U}(L)$, we have $\sum \alpha_i u_i \neq 0$ in

$\mathscr{U}(L)$. But, since $(f, g)_w \in J$, we have $\sum \alpha_i u_i = 0$ in $\mathscr{U}(L)$, which is a contradiction. ∎

Conversely, by Lemma 2.4 and the composition lemma, we can show that a Gröbner–Shirshov basis gives rise to a linear basis for the corresponding algebras.

THEOREM 2.10. (a) *If S is a Gröbner–Shirshov basis for the Lie superalgebra $L = \mathscr{L}_X/I$, then the set of S-reduced super-Lyndon–Shirshov monomials forms a linear basis of L.*

(b) *If S is a Gröbner–Shirshov basis for the universal enveloping algebra $\mathscr{U}(L) = \mathscr{A}_X/J$ of L, then the set of S-reduced words forms a linear basis of $\mathscr{U}(L)$.*

Proof. Since the proof of (b) is similar to that of (a), we prove (a) only. By Lemma 2.4 the set of S-reduced super-Lyndon–Shirshov monomials spans L. Assume that we have $\sum \alpha_i u_i = 0$ in L, where $\alpha_i \in k$ and u_i are distinct S-reduced super-Lyndon–Shirshov monomials. Then $\sum \alpha_i u_i \in I$ in the free Lie super algebra $\mathscr{L}_X$. Since $I \subset J$, we obtain $\sum \alpha_i u_i \in J$. By the composition lemma (Lemma 2.5) the leading term $\overline{\sum \alpha_i u_i}$ contains a subword $\bar{s}$ with $s \in S$. Since each u_i is S-reduced, we must have $\alpha_i = 0$ for all i. Hence the set of S-reduced super-Lyndon–Shirshov monomials is linearly independent. ∎

As a corollary, we obtain a purely combinatorial proof of the Poincaré–Birkhoff–Witt theorem.

PROPOSITION 2.11. *Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Lie superalgebra with a linear basis $Z = \{z_1, z_2, \dots\}$ such that each z_i is homogeneous with respect to the $\mathbb{Z}_2$-grading. Then a linear basis of the universal enveloping algebra $\mathscr{U}(L)$ of L is given by the set of all elements of the form $z_{i_1} z_{i_2} \cdots z_{i_n}$ where $i_k \leq i_{k+1}$ and $i_k \neq i_{k+1}$ if $z_{i_k} \in L_{\bar{1}}$.*

Proof. Let $Y = \{y_1, y_2, \dots\}$ be a $\mathbb{Z}_2$-graded set identified with the set Z by a map ι such that $\iota(y_i) = z_i$ and $\iota(Y_\alpha) = Z_\alpha$ with $\alpha \in \mathbb{Z}_2$. Let $\mathscr{L}_Y$ be the free Lie superalgebra generated by Y. Let $S \subset \mathscr{L}_Y$ be the set of elements of the form

$$[y_i y_j] - \sum_k \alpha_{ij}^k y_k,$$

where $i \geq j$ and $i \neq j$ if $y_i \in Y_{\bar{0}}$, and α_{ij}^k is the structure constants given by the equation $[z_i z_j] = \sum_k \alpha_{ij}^k z_k$ in L. Let I be the ideal of $\mathscr{L}_Y$ generated by S. Then, clearly, $\mathscr{L}_Y/I$ is isomorphic to L and the set of S-reduced super-Lyndon–Shirshov monomials is just the set Y. By Proposition 2.9 the set S is a Gröbner–Shirshov basis for L and then by Theorem 2.8 the set

S is also a Gröbner–Shirshov basis for $\mathscr{U}(L)$. Now our assertion follows from Theorem 2.10. ∎

Let S be a set of relations in the free Lie superalgebra $\mathscr{L}_X$ generated by X. We will see how one can *complete* the set S to get a Gröbner–Shirshov basis. For any subset T of $\mathscr{L}_X$, we define $\hat{T} = \{p/\alpha \mid \alpha \in k$ is the leading coefficient of $p \in T\}$. Let $S^{(0)} = \hat{S}$ and $S_{(0)} = \{\langle f, g\rangle_w \not\equiv_L 0 \bmod(S^{(0)}, w) \mid f, g \in S^{(0)}\}$. For $i \geq 1$, set $S_{(i)} = \{\langle f, g\rangle_w \not\equiv_L 0 \bmod(S^{(i)}, w) \mid f, g \in S^{(i)}\}$ and $S^{(i)} = S^{(i-1)} \cup \hat{S}_{(i-1)}$.

Then the set $S^c = \bigcup_{i \geq 0} S^{(i)}$ is a Gröbner–Shirshov basis for the (Lie) ideal I generated by S in $\mathscr{L}_X$. Hence, by Lemma 2.7, it is also a Gröbner–Shirshov basis for the (associative) ideal J generated by S in $\mathscr{A}_X$. It is easy to see that if every element of S is homogeneous in $x_i \in X$, then every element of S^c is also homogeneous in x_i's.

3. KAC–MOODY SUPERALGEBRAS

We now investigate the structure of Gröbner–Shirshov bases for Kac–Moody superalgebras. Our result is a generalization of the work by Bokut and Malcolmson [14] on the Gröbner–Shirshov bases for Kac–Moody algebras. In the section, since we consider the associative congruences only, we use the notation $\equiv$ in place of $\equiv_A$.

Let $\Omega = \{1, 2, \ldots, r\}$ be a finite index set and τ be a subset of Ω. A square matrix $A = (a_{ij})_{i,j \in \Omega}$ is called a *generalized Cartan matrix* if it satisfies:

(i) $a_{ii} = 2$ or 0 for $i = 1, \ldots, r$ and if $a_{ii} = 0$, then $i \in \tau$,

(ii) if $a_{ii} \neq 0$, then $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$,

(iii) $a_{ij} = 0$ implies $a_{ji} = 0$,

(iv) if $a_{ii} = 2$ and $i \in \tau$, then $a_{ij} \in 2\mathbb{Z}$.

Let $E = \{e_i\}_{i \in \Omega}$, $H = \{h_i\}_{i \in \Omega}$, $F = \{f_i\}_{i \in \Omega}$, and $X = E \cup H \cup F$. We define a $\mathbb{Z}_2$-grading on Ω by setting $\deg i = \bar{0}$ for $i \notin \tau$ and $\deg i = \bar{1}$ for $i \in \tau$, and on X by $\deg e_i = \deg f_i = \deg i$ and $\deg h_i = \bar{0}$. We give a linear ordering on X by $e_i \succ h_j \succ f_k$ for all $i, j, k \in \Omega$ and $e_i \succ e_j$, $h_i \succ h_j$, $f_i \succ f_j$ when $i > j$. Then we have the lexicographic ordering and the length-lexicographic ordering as in Section 2. We denote the left adjoint action of a Lie superalgebra by ad and the right adjoint action by $\widetilde{\text{ad}}$. The *Kac–Moody superalgebra* $\mathscr{G} = \mathscr{G}(A, \tau)$ *associated to* (A, τ) is defined to be the Lie superalgebra with generators X and the following defining relations,

$$
\begin{aligned}
&W: [h_i h_j] \qquad (i > j), \\
&\quad [e_i f_j] - \delta_{ij} h_i, \qquad [e_j h_i] + a_{ij} e_j, \qquad [h_i f_j] + a_{ij} f_j, \\
&S_{+,1}: (\operatorname{ad} e_i)^{1-n_{ij}} e_j \qquad (i > j), \\
&\quad e_i \left(\widetilde{\operatorname{ad}}\, e_j\right)^{1-n_{ji}} \qquad (i > j), \\
&S_{+,2}: [[e_{k+1}, e_k][e_k, e_{k-1}]] \qquad \text{for } k \in \eta, \\
&S_{-,1}: (\operatorname{ad} f_i)^{1-n_{ij}} f_j \qquad (i > j), \\
&\quad f_i \left(\widetilde{\operatorname{ad}} f_j\right)^{1-n_{ji}} \qquad (i > j), \\
&S_{-,2}: [[f_{k+1}, f_k][f_k, f_{k-1}]] \quad \text{for } k \in \eta,
\end{aligned}
\tag{3.1}
$$

where

$$
n_{ij} = \begin{cases} a_{ij} & \text{if } a_{ii} = 2 \text{ or } a_{ij} = 0 \\ -1 & \text{if } a_{ii} = 0 \text{ and } a_{ij} \neq 0 \end{cases} \quad \text{for } i \neq j, \tag{3.2}
$$

and η is the set of indices k such that $k \in \tau$, $k \pm 1 \notin \tau$, $a_{kk} = 0$, $a_{k+1,k-1} = 0$, and $a_{k,k+1} + a_{k,k-1} = 0$. Let $S_\pm = S_{\pm,1} \cup S_{\pm,2}$ and $S(A, \tau) = S_+ \cup W \cup S_-$. We denote by $\mathscr{G}_+$ (resp., $\mathscr{G}_0$ and $\mathscr{G}_-$) the subalgebra of $\mathscr{G}$ generated by E (resp., H and F).

Set $t_{ij} = [e_i f_j] - \delta_{i,j} h_i$, which belong to the relations W. We define the *differential substitution* $\tilde{\partial}_j = \tilde{\partial}(e_j \to h_j)$ acting as a right superderivation on $\mathscr{A}_E$ by

$$
\begin{aligned}
(e_i)\, \tilde{\partial}_j &= \delta_{ij} h_j, \\
(uv)\, \tilde{\partial}_j &= u(v)\, \tilde{\partial}_j + (-1)^{(\deg j)(\deg v)} (u)\, \tilde{\partial}_j v \quad \text{for } u, v \in \mathscr{A}_E.
\end{aligned}
\tag{3.3}
$$

It is easy to prove that for any $p \in \mathscr{A}_E$,

$$
p f_j \equiv (-1)^{(\deg p)(\deg j)} f_j p + (p)\, \tilde{\partial}_j \bmod (W, w) \tag{3.4}
$$

for some $w \gg \bar{p} f_j$. Note that $\tilde{\partial}_j$ is also a right superderivation on $\mathscr{L}_E$.

LEMMA 3.1. *Let p be a homogeneous monic element of $\mathscr{A}_E$ such that $(p, t_{ij})_w$ is defined for $w \in X^*$. Then we have*

$$
(p, t_{ij})_w \equiv (p)\, \tilde{\partial}_j \bmod (\{p\} \cup W, w).
$$

Proof. It suffices to consider the composition of intersection. We can write $p = \bar{p} + p'$ with $\bar{p} = be_i$, where all the terms of p' are lower than $\bar{p}$. Then $w = \bar{p}f_j = be_i f_j$. Since p is homogeneous, $\deg p = \deg p'$. From (3.4), we have

$$\begin{aligned}
(p, t_{ij})_w &= pf_j - b\Big(e_i f_j - (-1)^{(\deg i)(\deg j)} f_j e_i - \delta_{ij} h_j\Big) \\
&= p'f_j + (-1)^{(\deg i)(\deg j)} bf_j e_i + \delta_{ij} bh_j \\
&\equiv (-1)^{(\deg p)(\deg j)}\big(f_j p' + f_j be_i\big) + (p')\tilde{\partial}_j \\
&\quad + (-1)^{(\deg i)(\deg j)}(b)\tilde{\partial}_j e_i + \delta_{ij} bh_j \\
&\equiv (-1)^{(\deg p)(\deg j)} f_j p + (p)\tilde{\partial}_j \\
&\equiv (p)\tilde{\partial}_j \bmod(\{p\} \cup W, w).
\end{aligned}$$

∎

In the rest of this paper, we omit brackets whenever it is convenient. Namely, the Lie product $[a, b]$ is written as ab. Moreover, $(\operatorname{ad} x)^n y$ is written as $x^n y$ and $x(\widetilde{\operatorname{ad}} y)^n$ as xy^n. It would be clear from the context whether a product ab means a Lie product or not.

We write $f \equiv g \bmod(S, n)$ if $f - g = \sum \alpha_i a_i s_i b_i$ with $l(a_i \bar{s}_i b_i) \leq n$, where $n \in \mathbb{Z}_{>0}$, $\alpha_i \in k$, $a_i, b_i \in X^*$, and $s_i \in S$.

LEMMA 3.2. *Let $p \in S_+$. Then for any $l = 1, \ldots, r$, we have*

$$(p)\tilde{\partial}_l \equiv 0 \bmod\big(S_+ \cup W, l(\bar{p})\big).$$

Proof.

Case 1. Relation $S_{+,1}$:
Since $e_i^{1-n_{ij}} e_j = \alpha e_j e_i^{1-n_{ij}}$ with $\alpha \in k$, it suffices to prove our assertion for $p = e_j e_i^{1-n_{ij}}$ for $i \neq j$. We first consider the case when $a_{ii} = 2$. We have only to check the cases when $l = i$ and $l = j$. If $l = i$, we have

$$\begin{aligned}
(p)\tilde{\partial}_i &= \Big(e_j e_i^{1-a_{ij}}\Big)\tilde{\partial}_i \\
&= \big(e_j e_i^{-a_{ij}}\big)h_i + (-1)^{\deg i}\Big(\big(e_j e_i^{-a_{ij}-1}\big)h_i\Big)e_i \\
&\quad + (-1)^{2\deg i}\Big(\big(e_j e_i^{-a_{ij}-2}\big)h_i\Big)e_i^2 + \cdots + (-1)^{-a_{ij}\deg i}\big(e_j h_i\big)e_i^{-a_{ij}} \\
&\equiv a_{ij} e_j e_i^{-a_{ij}} + (-1)^{\deg i}\big(a_{ij} - 2\big)e_j e_i^{-a_{ij}} \\
&\quad + (-1)^{2\deg i}\big(a_{ij} - 4\big)e_j e_i^{-a_{ij}} + \cdots + (-1)^{-a_{ij}\deg i}(-a_{ij})\,e_j e_i^{-a_{ij}}.
\end{aligned}$$

If $i \notin \tau$, then, clearly, the coefficient of $e_j e_i^{-a_{ij}}$ is 0. If $i \in \tau$, then $a_{ij} \in 2\mathbb{Z}$ by the assumption on the generalized Cartan matrix A, and hence the coefficient of $e_j e_i^{-a_{ij}}$ is also 0.

Similarly, if $l = j$, we have

$$\begin{aligned}(p)\tilde{\partial}_j &= \left(e_j e_i^{1-a_{ij}}\right)\tilde{\partial}_j = (-1)^{(1-a_{ij})(\deg i)(\deg j)} h_j e_i^{1-a_{ij}} \\ &\equiv (-1)^{(1-a_{ij})(\deg i)(\deg j)} a_{ji} e_i e_i^{-a_{ij}} = 0.\end{aligned}$$

The proof for the case $a_{ii} = 0$ is the same.

Case 2. Relation $S_{+,2}$:

Let $p = (e_{k+1}e_k)(e_k e_{k-1})$ with $k \in \eta$. If $l = k-1$, since $(e_{k+1}e_k)e_k$ or $e_{k+1}e_k$ is in S_+, we have

$$\begin{aligned}(p)\tilde{\partial}_{k-1} &= ((e_{k+1}e_k)(e_k e_{k-1}))\tilde{\partial}_{k-1} = (e_{k+1}e_k)(e_k h_{k-1}) \\ &\equiv -a_{k-1,k}(e_{k+1}e_k)e_k \\ &\equiv 0 \bmod\left(S_+ \cup W, l(\bar{p})\right).\end{aligned}$$

Similarly, $(p)\tilde{\partial}_{k+1} \equiv 0 \bmod(S_+ \cup W, l(\bar{p}))$.

If $l = k$, since $a_{k,k-1} + a_{k,k+1} = 0$ and $e_{k+1}e_{k-1} \in S_+$, we have

$$\begin{aligned}(p)\tilde{\partial}_k &= ((e_{k+1}e_k)(e_k e_{k-1}))\tilde{\partial}_k \\ &= (e_{k+1}e_k)(h_k e_{k-1}) - (e_{k+1}h_k)(e_k e_{k-1}) \\ &\equiv a_{k,k-1}(e_{k+1}e_k)e_{k-1} + a_{k,k+1}e_{k+1}(e_k e_{k-1}) \\ &= (a_{k,k-1} + a_{k,k+1})e_{k+1}(e_k e_{k-1}) + a_{k,k-1}(e_{k+1}e_k)e_{k-1} \\ &\equiv 0 \bmod\left(S_+ \cup W, l(\bar{p})\right).\end{aligned}$$

∎

LEMMA 3.3. *For any element $p \in S_+^c$ and $j = 1, \ldots, r$, we have*

$$(p)\tilde{\partial}_j \equiv 0 \bmod\left(S_+^c \cup W, l(\bar{p})\right).$$

Proof. As we have seen in Section 2, we have $S_+^c = \bigcup S_+^{(i)}$ with $S_+^{(i)} \subset S_+^{(i+1)}$ for $i \geq 0$. Hence our assertion is equivalent to saying that if $p \in S_+^{(i)}$, then $(p)\tilde{\partial}_j \equiv 0 \bmod(S_+^{(i)} \cup W, l(\bar{p}))$ for each $i \geq 0$. We use induction on i. For $i = 0$, it is simply Lemma 3.2. Suppose that $(q)\tilde{\partial}_j \equiv 0 \bmod(S_+^{(i)} \cup W, l(\bar{q}))$ for all $q \in S_+^{(i)}$. Let $p \in S_+^{(i+1)} \setminus S_+^{(i)}$. Then $p = \langle q, r\rangle_w$ for some $q, r \in S_+^{(i)}$ and $\langle q, r\rangle_w \equiv (q, r)_w \bmod(S_+^{(i)}, w)$ by Lemma 2.7.

Since $l(w) = l(\bar{p})$, we have

$$\langle q, r\rangle_w \tilde{\partial}_j \equiv (q, r)_w \tilde{\partial}_j \bmod\big(S_+^{(i)} \cup W, l(\bar{p})\big).$$

Thus it is enough to show that $(q, r)_w \tilde{\partial}_j \equiv 0 \bmod(S_+^{(i)} \cup W, l(\bar{p}))$. Write $p = (q, r)_w = qa - br$. Then by the induction hypothesis, we have

$$\begin{aligned}(q, r)_w \tilde{\partial}_j &= q(a)\tilde{\partial}_j + (-1)^{(\deg a)(\deg j)}(q)\tilde{\partial}_j a - b(r)\tilde{\partial}_j \\ &\quad -(-1)^{(\deg r)(\deg j)}(b)\tilde{\partial}_j r \\ &\equiv 0 \bmod\big(S_+^{(i)} \cup W, l(\bar{p})\big).\end{aligned}$$

∎

Combining Lemmas 3.1 and 3.3, we obtain:

PROPOSITION 3.4. *For any element $p \in S_+^c$, we have*

$$\langle p, t_{ij}\rangle_w \equiv \big(p, t_{ij}\big)_w \equiv 0 \bmod(S_+^c \cup W, w).$$

Proposition 3.4 implies that all the compositions between the relations in S_+^c and W are trivial. Similarly, one can show that all the compositions between the relations in S_-^c and W are also trivial. Now we can present the main theorem of this section.

THEOREM 3.5. *Let $\mathcal{G} = \mathcal{G}(A, \tau)$ be a Kac–Moody superalgebra with the set of defining relations $S(A, \tau) = S_+ \cup W \cup S_-$. Then the set $S_+^c \cup W \cup S_-^c$ is a Gröbner–Shirshov basis for the Kac–Moody superalgebra $\mathcal{G}(A, \tau)$. That is, $S(A, \tau)^c = S_+^c \cup W \cup S_-^c$. Hence it is also a Gröbner–Shirshov basis for the universal enveloping algebra $\mathcal{U}(\mathcal{G})$ of $\mathcal{G}(A, \tau)$.*

Proof. By definition, there is no nontrivial composition among the relations in $S_\pm^c$ and the relations in S_+^c and S_-^c. Also, all the compositions between the relations between $S_\pm^c$ and W are trivial (see the remark after Proposition 3.4). Thus we have only to consider the compositions among the elements in W. We will show that $\langle p, q\rangle_w \equiv 0 \bmod(W, w)$ for all $p, q \in W$, where $w \in X^*$ is determined by p and q. There are four cases to be considered.

If $p = h_i h_j$ $(i > j)$ and $q = h_j h_k$ $(j > k)$, then $w = h_i h_j h_k$ and

$$\begin{aligned}\langle p, q\rangle_w &= [w]_p - [w]_q = (h_i h_j)h_k - h_i(h_j h_k) \\ &= (h_i h_k)h_j \equiv 0 \bmod(W, w).\end{aligned}$$

If $p = e_j h_i + \alpha_{ij} e_j$ and $q = h_i h_k$ $(i > k)$, then $w = e_j h_i h_k$ and

$$\begin{aligned}\langle p, q\rangle_w &= [w]_p - [w]_q = (e_j h_i) h_k + a_{ij} e_j h_k - e_j(h_i h_k)\\ &= (e_j h_k) h_i + a_{ij} e_j h_k \equiv -a_{kj} e_j h_i + a_{ij} e_j h_k\\ &\equiv a_{kj} a_{ij} e_j - a_{kj} a_{ij} e_j = 0 \bmod(W, w).\end{aligned}$$

Similarly, if $p = h_i h_j$ $(i > j)$ and $q = h_j f_k + a_{jk} f_k$, then $\langle p, q\rangle_w \equiv 0 \bmod(W, w)$. Finally, if $p = e_j h_i + a_{ij} e_j$ and $q = h_i f_k + a_{ik} f_k$, then $w = e_j h_i f_k$ and

$$\begin{aligned}\langle p, q\rangle_w &= [w]_p - [w]_q = (e_j h_i) f_k + a_{ij} e_j f_k - e_j(h_i f_k) - a_{ik} e_j f_k\\ &= (e_j f_k) h_i + a_{ij} e_j f_k - a_{ik} e_j f_k\\ &\equiv \delta_{jk} h_j h_i + \delta_{jk} a_{ij} h_j - \delta_{jk} a_{ik} h_j \equiv 0 \bmod(W, w),\end{aligned}$$

which completes the proof. ∎

As a corollary, we obtain the *triangular decomposition* of Kac–Moody superalgebras and their universal enveloping algebras.

COROLLARY 3.6. *Let $\mathscr{G} = \mathscr{G}(A, \tau)$ be a Kac–Moody superalgebra. Then we have*

$$\mathscr{G} \cong \mathscr{G}_+ \oplus \mathscr{G}_0 \oplus \mathscr{G}_- \tag{3.5}$$

and

$$U(\mathscr{G}) \cong U(\mathscr{G}_+) \otimes U(\mathscr{G}_0) \otimes U(\mathscr{G}_-) \tag{3.6}$$

as k-linear spaces.

Proof. Observe that any super-Lyndon–Shirshov monomial of degree ≥ 2 cannot be W-reduced if it contains h_i or $e_j f_k$ as a subword. Hence by Theorem 3.5, the set B of $S(A, \tau)^c$-reduced super-Lyndon–Shirshov monomials is given by $B = B_+ \cup H \cup B_-$, where B_+ (resp., B_-) is the set of S_+^c-reduced (resp., S_-^c-reduced) super-Lyndon–Shirshov monomials in e_i's (resp., f_i's). By Theorem 2.10, B is a linear basis of $\mathscr{G}$, which proves the k-linear isomorphism (3.5). The isomorphism (3.6) follows from the Poincaré–Birkhoff–Witt theorem. ∎

4. CLASSICAL LIE SUPERALGEBRAS

In this section, we give an explicit construction of Gröbner–Shirshov bases for the classical Lie superalgebras. A Gröbner–Shirshov basis S is said to be *minimal* if no proper subset of S is closed under the Lie composition. We first set up some notations. Recall that we omit brackets whenever it is convenient. For the elements $x_i \in X$, we set $[x_1 x_2 \cdots x_m] = x_1[x_2 \cdots x_m]$ and $\{x_1 \cdots x_{m-1}x_m\} = \{x_1 \cdots x_{m-1}\}x_m$ $(m \geq 1)$. If $i > j$, we write $x_{ij} = [x_i x_{i-1} \cdots x_j]$. For simplicity, we also denote $x_{ii} = x_i$. We use the lexicographical ordering for the set $\Omega \times \Omega$: $(i, j) > (k, l)$ if and only if $i > k$ or $i = k$, $j > l$.

We briefly recall the definition of classical Lie superalgebras [21]. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space with $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$, and let L be the space of k-linear endomorphisms of V. For each $\alpha \in \mathbb{Z}_2$, set

$$L_\alpha = \left\{T\colon V \to V \mid T\left(V_\beta\right) \subset V_{\alpha+\beta} \text{ for all } \beta \in \mathbb{Z}_2\right\}.$$

Then L has a $\mathbb{Z}_2$-graded decomposition $L = L_{\bar{0}} \oplus L_{\bar{1}}$ and it becomes a Lie superalgebra with the superbracket defined by

$$[X, Y] = XY - (-1)^{\alpha\beta} YX$$

for $X \in L_\alpha$, $Y \in L_\beta$, $\alpha, \beta \in \mathbb{Z}_2$. The Lie superalgebra L is called the *general linear Lie superalgebra* and is denoted by $gl(m, n)$.

Let $v_1, \ldots, v_m$ be a basis of $V_{\bar{0}}$ and $v_{m+1}, \ldots, v_{m+n}$ be a basis of $V_{\bar{1}}$. Then L can be interpreted as the space of $(m+n) \times (m+n)$ matrices over k, and we have

$$L_{\bar{0}} = \left\{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| A \text{ is an } m \times m \text{ matrix and } D \text{ is an } n \times n \text{ matrix}\right\},$$

$$L_{\bar{1}} = \left\{\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \middle| B \text{ is an } m \times n \text{ matrix and } C \text{ is an } n \times m \text{ matrix}\right\}.$$

For

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in gl(m, n),$$

we define the *supertrace* of X to be $\operatorname{str} X = \operatorname{tr} A - \operatorname{tr} B$, where tr denotes the usual trace function. Then the subspace $sl(m, n)$ of $gl(m, n)$ consisting of the matrices with supertrace 0 forms a Lie superalgebra which is called the *special linear Lie superalgebra*.

Let B be a nondegenerate consistent supersymmetric bilinear form on V. Thus $V_{\bar{0}}$ and $V_{\bar{1}}$ are orthogonal to each other, $B|_{V_{\bar{0}} \times V_{\bar{0}}}$ is symmetric, and $B|_{V_{\bar{1}} \times V_{\bar{1}}}$ is skew-symmetric (which implies n must be even). For each

$\alpha \in \mathbb{Z}_2$, define

$$\begin{aligned}&\operatorname{osp}(m,n)_\alpha \\ &\quad = \{T \in gl(m,n)_\alpha | B(Tv,w) \\ &\qquad = -(-1)^{\alpha(\deg v)} B(v,Tw) \text{ for all } v,w \in V\}.\end{aligned}$$

Then the subspace $\operatorname{osp}(m,n) = \operatorname{osp}(m,n)_{\bar{0}} \oplus \operatorname{osp}(m,n)_{\bar{1}}$ becomes a Lie superalgebra. We set

$$\begin{aligned} B(m,n) &= \operatorname{osp}(2m+1,2n) \qquad (m \geq 0, n > 0), \\ C(n) &= \operatorname{osp}(2,2n-2) \qquad (n \geq 2), \\ D(m,n) &= \operatorname{osp}(2m,2n) \qquad (m \geq 2, n > 0). \end{aligned} \tag{4.1}$$

These subalgebras are called the *ortho-symplectic Lie superalgebras of type* $B(m,n)$, $C(n)$, and $D(m,n)$, respectively.

4.1. *The Special Linear Lie Superalgebra* $sl(m,n)$ $(m,n > 0)$

Let E_{ij} denote the $(m+n) \times (m+n)$ matrix whose (i,j)-entry is equal to 1 and all the other entries are 0, and let

$$x_i = E_{i,i+1}, \quad y_i = E_{i+1,i} \qquad (i = 1,2,\dots,m+n-1). \tag{4.2}$$

Then the elements $x_i, y_i, z_i = [x_i, y_i]$ $(i = 1,2,\dots,m+n-1)$ generate the Lie superalgebra $sl(m,n)$.

On the other hand, let $\Omega = \{1,2,\dots,m+n-1\}$, $\tau = \{m\} \subset \Omega$, and consider the generalized Cartan matrix $A = (a_{ij})_{i,j\in\Omega}$ defined by

$$\begin{aligned} &a_{m,m} = 0, \qquad a_{m,m+1} = 1, \qquad a_{m+1,m} = -1, \\ &a_{ij} = -1 \quad \text{if } |i-j| = 1 \quad \text{and} \quad (i,j) \neq (m,m+1), \\ &a_{ij} = 0 \quad \text{if } |i-j| > 1. \end{aligned} \tag{4.3}$$

Let $\mathscr{G} = \mathscr{G}(A,\tau)$ be the Kac–Moody superalgebra associated with (A,τ) and denote by e_i, f_i, h_i $(i = 1,\dots,m+n-1)$ the generators of $\mathscr{G}$. Then it is straightforward to verify that the generators x_i, y_i, z_i $(i = 1,\dots,m+n-1)$ of the Lie superalgebra $sl(m,n)$ also satisfy the defining relations of the Kac–Moody algebra $\mathscr{G} = \mathscr{G}(A,\tau)$. Hence there exists a surjective Lie superalgebra homomorphism $\phi\colon \mathscr{G} \to sl(m,n)$ given by $e_i \mapsto x_i$, $f_i \mapsto y_i$, $h_i \mapsto z_i$ $(i = 1,2,\dots,m+n-1)$.

In the following lemma, we derive more "refined" relations of $\mathscr{G}$, which are used to construct a Gröbner–Shirshov basis for the special linear Lie superalgebra $sl(m, n)$. Recall that we use the notation $e_{ij} = [e_i e_{i-1} \cdots e_j]$ for $i > j$ and $e_{ii} = e_i$.

LEMMA 4.1. *In the Kac–Moody superalgebra $\mathscr{G} = \mathscr{G}(A, \tau)$, we have*

$$e_{ij} e_{kl} = \delta_{j-1,k} e_{il} \quad \textit{for all } (i, j) \geq (k, l). \tag{4.4}$$

Proof. We proceed in several steps.

Step 1. For all $j > k + 1$, we have $e_{ij} e_{kl} = 0$.

By the Serre relations, we have $e_j e_l = 0$ for all $j > l + 1$. Next, fix l and assume that $j > k + 1$, $k > l$. Then by the Jacobi identity and induction hypothesis, we get

$$e_j e_{kl} = e_j(e_k e_{k-1,l}) = (e_j e_k) e_{k-1,l} + (-1)^d e_k(e_j e_{k-1,l}) = 0,$$

where $d = (\deg e_j)(\deg e_k) \in \mathbb{Z}_2$. Finally, fix j and assume that $i > j > k + 1$. Then the induction argument yields

$$e_{ij} e_{kl} = (e_i e_{i-1,j}) e_{kl} = e_i(e_{i-1,j} e_{kl}) + (-1)^d (e_i e_{kl}) e_{i-1,j} = 0,$$

where $d = (\deg e_i)(\deg e_{i-1}) \in \mathbb{Z}_2$.

Step 2. For all $i, j, k \in \Omega$, we have $e_{ij} e_{j-1,k} = e_{ik}$.

If $i = j$, there is nothing to prove. If $i > j$, then by induction argument and Step 1, we obtain

$$\begin{aligned} e_{ij} e_{j-1,k} &= (e_i e_{i-1,j}) e_{j-1,k} = e_i(e_{i-1,j} e_{j-1,k}) + (-1)^d (e_i e_{j-1,k}) e_{i-1,j} \\ &= e_{ii} e_{i-1,k} = e_{ik}, \end{aligned}$$

where $d = (\deg e_i)(\deg e_{i-1}) \in \mathbb{Z}_2$.

Step 3. For all $i > j$, we have $e_i e_{ij} = 0$ and $e_{ij} e_j = 0$.

By the Serre relations, we have $e_i e_{i,i-1} = 0$. If $i \geq j + 2$, then Step 2 implies $e_{ij} = e_{i,i-1} e_{i-2,j}$. Hence by Step 1, we obtain

$$e_i e_{ij} = e_i(e_{i,i-1} e_{i-2,j}) = (e_i e_{i,i-1}) e_{i-2,j} + (-1)^d e_{i,i-1}(e_i e_{i-2,j}) = 0,$$

where $d = (\deg e_i)(\deg e_{i,i-1}) \in \mathbb{Z}_2$.

Similarly, we get $e_{ij} e_j = 0$ for $i > j$.

Step 4. For all $k, l \geq 1$, we have $h_i e_{i+k,i-l} = 0$.

By the relations in W, we obtain

$$h_i e_{i+k,i-l} = (a_{i,i+1} + a_{ii} + a_{i,i-1}) e_{i+k,i-l} = 0.$$

Step 5. For all $i > j$, we have $e_{ij}e_{i-1} = 0$.

If $j = i - 1$, then by the Serre relations, we get $e_{ij}e_{i-1} = 0$. Suppose first that $j < i - 1$ and $i - 1 \neq m$. Then by Step 3, we obtain

$$\begin{aligned}(e_{ij}e_{i-1})e_{i-1} &= \big((e_i e_{i-1,j})e_{i-1}\big)e_{i-1}\\ &= \big(e_i(e_{i-1,j}e_{i-1}) + (-1)^d(e_ie_{i-1})e_{i-1,j}\big)e_{i-1}\\ &= (-1)^d(e_ie_{i-1})(e_{i-1,j}e_{i-1}) + (-1)^{d'}\big((e_ie_{i-1})e_{i-1}\big)e_{i-1,j}\\ &= 0,\end{aligned}$$

where $d = (\deg e_i)(\deg e_{i-1})$ and $d' = (\deg e_{i-1})(\deg e_{i-1,j})$. Multiplying both sides by f_{i-1} yields

$$\begin{aligned}0 &= \big((e_{ij}e_{i-1})e_{i-1}\big)f_{i-1}\\ &= (e_{ij}e_{i-1})(e_{i-1}f_{i-1}) + \big((e_{ij}e_{i-1})f_{i-1}\big)e_{i-1}\\ &= (e_{ij}e_{i-1})h_{i-1} + (e_{ij}h_{i-1})e_{i-1} + \big((e_{ij}f_{i-1})e_{i-1}\big)e_{i-1}.\end{aligned}$$

The second summand is equal to 0 by Step 4. Since $e_{ij}f_{i-1}$ is a scalar multiple of $e_ie_{i-2,j}$, the third summand is also equal to 0. By the Jacobi identity and Step 4, the first summand yields $2e_{ij}e_{i-1} = 0$, which proves our claim.

If $j < i - 1$ and $i - 1 = m$, since $(e_{m+1}e_m)(e_me_{m-1}) = 0$ by the Serre relations, we get

$$\begin{aligned}&e_{m+1,j}e_m\\ &\quad= e_{m+1}(e_{mj}e_m) - (e_{m+1}e_m)e_{mj}\\ &\quad= -(e_{m+1}e_m)\big(e_m(e_{m-1}e_{m-2,j})\big)\\ &\quad= -(e_{m+1}e_m)\big((e_me_{m-1})e_{m-2,j}\big)\\ &\quad= -\big((e_{m+1}e_m)(e_me_{m-1})\big)e_{m-2,j} + (e_me_{m-1})\big((e_{m+1}e_m)e_{m-2,j}\big)\\ &\quad= 0.\end{aligned}$$

Step 6. For all $n > k \geq 0$, $m > l \geq 0$, we have $e_{m+k,m-l}e_{m+k,m-l} = 0$.

Suppose $k = 0$. If $l = 0$, then we have to show that $e_me_m = 0$. Note that

$$0 = e_m(e_me_{m-1}) = (e_me_m)e_{m-1} - e_m(e_me_{m-1}) = (e_me_m)e_{m-1}.$$

Multiplying both sides by f_{m-1}, we obtain

$$
\begin{aligned}
0 &= \big((e_m e_m) e_{m-1}\big) f_{m-1} \\
&= (e_m e_m)(e_{m-1} f_{m-1}) + \big((e_m e_m) f_{m-1}\big) e_{m-1} \\
&= (e_m e_m) h_{m-1} = 2 e_m e_m,
\end{aligned}
$$

which implies $e_m e_m = 0$.

Next, suppose $l > 0$. If $e_{m,m-l} e_{m,m-l} = 0$, then

$$
\begin{aligned}
0 &= \big((e_{m,m-l} e_{m,m-l}) e_{m-l-1}\big) e_{m-l-1} \\
&= (e_{m,m-l} e_{m,m-l-1}) e_{m-l-1} + (e_{m,m-l-1} e_{m,m-l}) e_{m-l-1} \\
&= 2 e_{m,m-l-1} e_{m,m-l-1},
\end{aligned}
$$

which yields $e_{m,m-l-1} e_{m,m-l-1} = 0$. Hence, by the downward induction, we conclude $e_{m,m-l} e_{m,m-l} = 0$ for all $m > l \geq 0$.

Finally, if $k > 0$, then our assertion follows from the same downward induction argument as above.

Step 7. For all $k \geq k'$, $l \leq l'$, we have $e_{m+k,m-l} e_{m+k',m-l'} = 0$.

Suppose $k' = k$. If $l = l'$, then our assertion was proven in Step 6. If $l < l'$ and $e_{m+k,m-l} e_{m+k,m-l'} = 0$, then

$$
\begin{aligned}
0 &= (e_{m+k,m-l} e_{m+k,m-l'}) e_{m-l'-1} \\
&= e_{m+k,m-l} (e_{m+k,m-l'-1}) + (e_{m+k,m-l} e_{m-l'-1}) e_{m+k,m-l'} \\
&= e_{m+k,m-l} e_{m+k,m-l'-1}.
\end{aligned}
$$

Hence by the downward induction, we get $e_{m+k,m-l} e_{m+k,m-l'} = 0$ for all $l \leq l'$.

If $k > k'$, our assertion follows by the same downward induction argument.

Step 8. For all $i \geq j > 1$, we have $e_{ij} e_{i,j-1} = 0$.

If $i = j$, then our assertion is just the Serre relation. Suppose $i > j$ and $i + 1 \neq m$. Then if $e_{ij} e_{i,j-1} = 0$, we have

$$
\begin{aligned}
0 &= e_{i+1}\big(e_{i+1}(e_{ij} e_{i,j-1})\big) \\
&= e_{i+1}(e_{i+1,j} e_{i,j-1}) + (-1)^d e_{i+1}(e_{ij} e_{i+1,j-1}) \\
&= (-1)^{d'} e_{i+1,j} e_{i+1,j-1} + (-1)^d e_{i+1,j} e_{i+1,j-1},
\end{aligned}
$$

where $d = (\deg e_{i+1})(\deg e_{ij})$ and $d' = (\deg e_{i+1})(\deg e_{i+1,j})$. Since $i + 1 \neq m$, we have $e_{i+1,j} e_{i+1,j-1} = 0$ and the induction argument gives our relations. If $i > j$, $i+1 = m$, and $e_{ij}e_{i,j-1} = 0$, then by Step 7, we get $e_{i+1,j}e_{i+1,j-1} = e_{mj}e_{m,j-1} = 0$. Hence our assertion follows from the induction.

Step 9. For all $k \neq j-1$, $(i,j) \geq (k,l)$, we have $e_{ij}e_{kl} = 0$.

Fix $k = i$. If $l = j$, then our assertion holds by Step 6. If $l = j-1$, then it is just Step 8. If $l < j-1$, then, by Steps 1 and 8, we have

$$\begin{aligned} e_{ij}e_{il} &= e_{ij}(e_{i,j-1}e_{j-2,l}) \\ &= (e_{ij}e_{i,j-1})e_{j-2,l} + (-1)^d e_{i,j-1}(e_{ij}e_{j-2,l}) = 0, \end{aligned}$$

where $d = (\deg e_{ij})(\deg e_{i,j-1}) \in \mathbb{Z}_2$.

Suppose $k < i$. If $j > k+1$, our assertion holds by Step 1. Let us assume $k \geq j$. If $k = l$, then we may assume $k < i-1$ by Step 5, and we have

$$\begin{aligned} e_{ij}e_k &= (e_{i,k+2}e_{k+1,j})e_k \\ &= e_{i,k+2}(e_{k+1,j}e_k) + (-1)^d (e_{i,k+2}e_k)e_{k+1,j} = 0. \end{aligned}$$

We use induction on $k - l$. Note that if $k > l$, then we have

$$e_{ij}e_{kl} = e_{ij}(e_k e_{k-1,l}) = (e_{ij}e_k)e_{k-1,l} + (-1)^d e_k(e_{ij}e_{k-1,l}),$$

where $d = (\deg e_{ij})(\deg e_k) \in \mathbb{Z}_2$. The first summand is equal to 0 by the case $k = l$. Consider the second summand. If $j \neq k$, then it is 0 by the induction hypothesis. If $j = k$, then by Step 2, it is equal to

$$(-1)^d e_k(e_{ik}e_{k-1,l}) = (-1)^d e_k e_{il} = 0.$$

∎

Let $X = E \cup H \cup F = \{e_i, h_i, f_i \mid i \in \Omega\}$ be a $\mathbb{Z}_2$-graded set, where $\Omega = \{1, 2, \ldots, m+n-1\}$ and $\tau = \{m\}$ is the set of odd index. Let R_+ be the set of relations in $E^\#$ given by:

I. e_ie_j $(i > j+1)$,

II. $e_{ij}e_{i-1}$ $(i > j)$,

III. $e_{ij}e_{i,j-1}$ $(i \geq j > 1)$,

IV. $e_{m+k,m-l}e_{m+k,m-l}$ $(n > k \geq 0,\ m > l \geq 0)$.

Let R_- be the set of relations in $F^\#$ obtained by replacing e_{ij}'s in R_+ by f_{ij}'s, and let $R(A,\tau) = R_+ \cup W \cup R_-$. Consider the Lie superalgebra $L = \mathscr{L}_X / \langle R(A,\tau) \rangle$, where $\langle R(A,\tau) \rangle$ denotes the ideal in $\mathscr{L}_X$ generated by $R(A,\tau)$. Then, by Lemma 4.1, there is a surjective Lie superalgebra homomorphism $\psi\colon L \to \mathscr{G}$ defined by $e_i \mapsto e_i$, $h_i \mapsto h_i$, $f_i \mapsto f_i$ $(i \in \Omega)$. We now prove the main result of this subsection.

THEOREM 4.2. *The set $R(A, \tau)$ of relations in $\mathscr{L}_X$ is a Gröbner–Shirshov basis for the Lie superalgebra L.*

Proof. Set $R = R(A, \tau)$. As in the proof of Corollary 3.6, the set of $R(A, \tau)$-reduced super-Lyndon–Shirshov monomials is $B = B_+ \cup H \cup B_-$, where $B_\pm$ is the set of $R_\pm$-reduced super-Lyndon–Shirshov monomials in $\mathscr{L}_E$ (resp., in $\mathscr{L}_F$). We claim that the set of R_+-reduced Lyndon–Shirshov monomials in $\mathscr{L}_E$ is

$$B'_+ = \left\{e_{ij} \middle| m + n > i \geq j \geq 1\right\}.$$

Let w be an R_+-reduced Lyndon–Shirshov monomial in $\mathscr{L}_E$. If $l(w) = 1$, then there is nothing to prove. Suppose that $l(w) > 1$. Then $w = uv$, where u, v are R_+-reduced Lyndon–Shirshov monomials. By induction, we have $w = e_{ij}e_{kl}$, where $i \geq j$, $k \geq l$, and $(i, j) > (k, l)$ in the lexicographical ordering. Note that we must have $i > k$, for if $i = k$, then $j - 1 \geq l$ and $\overline{e_{ij}e_{i, j-1}}$ is a subword of $\overline{w}$. We show that $k = j - 1$ and $i = j$. If $k > j$, then $\overline{w}$ contains $\overline{e_{k+1, j}e_k}$ as a subword, and if $k = j$, then $\overline{w}$ contains $\overline{(e_{k+1}e_k)e_k}$ as a subword. Finally, if $k \leq j - 2$, then $\overline{w}$ contains $\overline{e_j e_k}$ as a subword. Hence we must have $k = j - 1$. Moreover, since w is a Lyndon–Shirshov monomial, we must have $i = j$. Therefore, we obtain $w = e_{il}$, which proves our claim.

Now, let w be an R_+-reduced super-Lyndon–Shirshov monomial in $\mathscr{L}_E$. Then w is a Lyndon–Shirshov monomial or $w = uu$ with u a Lyndon–Shirshov monomial in $E_1^{\#}$. If the latter is true, then, as we have seen in the previous paragraph, we have $u = e_{m+k, m-l}$ $(n > k \geq 0$, $m > l \geq 0)$, in which case w is not R_+-reduced by IV. Therefore we have

$$B_+ = B'_+ = \left\{e_{ij} \middle| m + n > i \geq j \geq 1\right\}.$$

Similarly, we get $B_- = \{f_{ij} \mid m + n > i \geq j \geq 1\}$.

By Lemma 2.4, B spans L. Since ϕ and ψ are surjective, we have $\operatorname{card}(B) \geq \dim sl(m, n)$. But the number of elements of B is $(m + n)^2 - 1$, which is equal to the dimension of $sl(m, n)$. Thus ϕ and ψ are isomorphisms and B is a linear basis of L. Therefore, by Proposition 2.9, R is a Gröbner–Shirshov basis for L. ∎

Remark. The proof of Theorem 4.2 shows that the Lie superalgebras $L, \mathscr{G}(A, \tau)$ and $sl(m, n)$ are all isomorphic. Hence Theorem 4.2 gives a Gröbner–Shirshov basis for the Lie superalgebra $sl(m, n)$. Our argument also shows that $R(A, \tau)$ is actually a minimal Gröbner–Shirshov basis.

4.2. *The Lie Superalgebras of Type* $B(m,n)$ $(m,n>0)$

Let E_{ij} denote the $(2m+2n+1)\times(2m+2n+1)$ matrix whose (i,j)-entry is 1 and all the other entries are 0. Set

$$\begin{aligned}
x_i &= E_{2m+i+1,2m+i+2} - E_{2m+n+i+2,2m+n+i+1} \qquad (1\le i\le n-1),\\
x_n &= E_{2m+n+1,1} + E_{m+1,2m+2n+1},\\
x_{n+i} &= E_{i,i+1} - E_{m+i+1,m+i} \qquad (1\le i\le m-1),\\
x_{m+n} &= \sqrt{2}\,(E_{m,2m+1} - E_{2m+1,2m}), \qquad (4.5)\\
y_i &= E_{2m+i+2,2m+i+1} - E_{2m+n+i+1,2m+n+i+2} \qquad (1\le i\le n-1),\\
y_n &= E_{1,2m+n+1} - E_{2m+2n+1,m+1},\\
y_{n+i} &= E_{i+1,i} - E_{m+i,m+i+1} \qquad (1\le i\le m-1),\\
y_{m+n} &= \sqrt{2}\,(E_{2m+1,m} - E_{2m,2m+1}).
\end{aligned}$$

Then the elements $x_i, y_i, z_i = [x_i, y_i]$ $(i = 1,2,\dots,m+n)$ generate the ortho-symplectic Lie superalgebra $B(m,n) = \mathrm{osp}(2m+1,2n)$ $(m,n>0)$ and x_n, y_n are the odd generators.

On the other hand, let $\Omega = \{1,2,\dots,m+n\}$, $\tau = \{n\} \subset \Omega$, and consider the generalized Cartan matrix $A = (a_{ij})_{i,j\in\Omega}$ defined by

$$\begin{aligned}
&a_{n,n} = 0, \qquad a_{n,n+1} = 1, \qquad a_{m+n,m+n-1} = -2,\\
&a_{ij} = -1 \quad \text{if } |i-j| = 1, (i,j) \ne (n,n+1), (m+n,m+n-1),\\
&a_{ij} = 0 \quad \text{if } |i-j| > 1. \qquad (4.6)
\end{aligned}$$

Let $\mathscr{G} = \mathscr{G}(A,\tau)$ be the Kac–Moody superalgebra associated with (A,τ) and denote by e_i, f_i, h_i $(i = 1,2,\dots,m+n)$ the generators of $\mathscr{G}$. Then, as in the case of $sl(m,n)$, one can verify that the generators x_i, y_i, z_i $(i = 1,2,\dots,m+n)$ of the Lie superalgebra $\mathrm{osp}(2m+1,2n)$ satisfy the defining relations of the Kac–Moody superalgebra $\mathscr{G}(A,\tau)$. Hence there exists a surjective Lie superalgebra homomorphism $\phi\colon \mathscr{G} \to \mathrm{osp}(2m+1,2n)$ given by $e_i \mapsto x_i$, $f_i \mapsto y_i$, $h_i \mapsto z_i$ $(i = 1,2,\dots,m+n)$. As in Section 4.1, we first derive more relations in $\mathscr{G}$, which are used to construct a Gröbner–Shirshov basis for the ortho-symplectic Lie superalgebra $B(m,n) = \mathrm{osp}(2m+1,2n)$ $(m,n>0)$.

LEMMA 4.3. *In the Kac–Moody superalgebra* $\mathscr{G} = \mathscr{G}(A,\tau)$, *we have*

$$\begin{aligned}
e_{ij}e_{kl} &= \delta_{j-1,k}e_{il} \quad \text{if } (i,j) \ge (k,l),\ m+n > k,\\
[e_{m+n,i}e_{m+n,j}e_{m+n,k}] &= 0 \qquad (i,j,k\in\Omega), \qquad (4.7)\\
(e_{m+n,i}e_{m+n,j})(e_{m+n,k}e_{m+n,l}) &= 0 \qquad (i,j,k,l\in\Omega).
\end{aligned}$$

Proof. As in Lemma 4.1, we will prove our assertion in several steps.

Step 1. For all $(i,j) > (k,l)$ and $m+n > k$, we have $e_{ij}e_{kl} = \delta_{j-1,k}e_{il}$.

If we remove the $(m+n)$th row and the $(m+n)$th column of A, then we get the generalized Cartan matrix for the Lie superalgebra $sl(m,n)$. Thus we have only to consider the case when $i = m+n$. Suppose $k \leq m + n - 2$. If $j = m+n$, then $e_{m+n}e_{kl} = 0$ as in Step 1 of the proof of Lemma 4.1. If $j < m+n$, then we have

$$\begin{aligned} e_{m+n,j}e_{kl} &= (e_{m+n}e_{m+n-1,j})e_{kl} \\ &= e_{m+n}(e_{m+n-1,j}e_{kl}) + (-1)^d(e_{m+n}e_{kl})e_{m+n-1,j} \\ &= \delta_{j-1,k}e_{m+n}e_{m+n-1,l} = \delta_{j-1,k}e_{m+n,l}, \end{aligned}$$

where $d = (\deg e_{m+n})(\deg e_{m+n-1,j})$.

If $k = m+n-1$ and $j = m+n$, then $e_{m+n}e_{m+n-1,l} = e_{m+n,l}$ and if $j < m+n-1$, then

$$\begin{aligned} &e_{m+n,j}e_{m+n-1,l} \\ &\quad = e_{m+n,j}(e_{m+n-1}e_{m+n-2,l}) \\ &\quad = (e_{m+n,j}e_{m+n-1})e_{m+n-2,l} + (-1)^d e_{m+n-1}(e_{m+n,j}e_{m+n-2,l}) \\ &\quad = (e_{m+n,j}e_{m+n-1})e_{m+n-2,l} + (-1)^d \delta_{j-1,m+n-2}e_{m+n-1}e_{m+n,l}, \end{aligned}$$

where $d = (\deg e_{m+n,j})(\deg e_{m+n-1})$. As in Step 5 of the proof of Lemma 4.1, we have $e_{m+n,j}e_{m+n-1} = 0$, which proves our claim.

Step 2. For all $i \in \Omega$, we have $[e_{m+n}e_{m+n}e_{m+n,i}] = 0$.

It is clear that $[e_{m+n}e_{m+n}e_{m+n}] = 0$. Suppose that $[e_{m+n}e_{m+n}e_{m+n,i}] = 0$ for $i < m+n$. Multiplying both sides by e_{i-1}, we obtain

$$\begin{aligned} 0 &= [e_{m+n}e_{m+n}e_{m+n,i}]e_{i-1} \\ &= e_{m+n}((e_{m+n}e_{m+n,i})e_{i-1}) + (-1)^d(e_{m+n}e_{i-1})(e_{m+n}e_{m+n,i}) \\ &= e_{m+n}(e_{m+n}e_{m+n,i-1}) + (-1)^{d'}e_{m+n}((e_{m+n}e_{i-1})e_{m+n,i}) \\ &= [e_{m+n}e_{m+n}e_{m+n,i-1}], \end{aligned}$$

for $d, d' \in \mathbb{Z}_2$, and the downward induction on i gives our claim.

Step 3. $[e_{m+n}e_{m+n}e_{m+n-1}](e_{m+n}e_{m+n-1}) = 0$.
If $m \neq 1$, then by the Serre relation, we get

$$\begin{aligned}
&[e_{m+n}e_{m+n}e_{m+n-1}](e_{m+n}e_{m+n-1})\\
&\quad = ([e_{m+n}e_{m+n}e_{m+n-1}]e_{m+n})e_{m+n-1}\\
&\qquad + e_{m+n}([e_{m+n}e_{m+n}e_{m+n-1}]e_{m+n-1})\\
&\quad = -[e_{m+n}e_{m+n}e_{m+n}e_{m+n-1}]e_{m+n-1}\\
&\qquad + e_{m+n}(e_{m+n}\{e_{m+n}e_{m+n-1}e_{m+n-1}\}) = 0.
\end{aligned}$$

If $m = 1$, then

$$\begin{aligned}
&[e_{n+1}e_{n+1}e_n](e_{n+1}e_n)\\
&\quad = ([e_{n+1}e_{n+1}e_n]e_{n+1})e_n + e_{n+1}([e_{n+1}e_{n+1}e_n]e_n)\\
&\quad = e_{n+1}(e_{n+1}\{e_{n+1}e_n e_n\}) - e_{n+1}((e_{n+1}e_n)(e_{n+1}e_n))\\
&\quad = -[e_{n+1}e_{n+1}e_n](e_{n+1}e_n) - (e_{n+1}e_n)[e_{n+1}e_{n+1}e_n]\\
&\quad = -2[e_{n+1}e_{n+1}e_n](e_{n+1}e_n),
\end{aligned}$$

which yields $[e_{n+1}e_{n+1}e_n][e_{n+1}e_n] = 0$.

Step 4. For all $i \in \Omega$, we have

$$[e_{m+n}e_{m+n}e_{m+n-1}]e_{m+n,i} = 0, \qquad (e_{m+n}e_{m+n,i})(e_{m+n}e_{m+n-1}) = 0.$$

Let $a = [e_{m+n}e_{m+n}e_{m+n-1}]e_{m+n,i}$ and $b = (e_{m+n}e_{m+n,i})(e_{m+n}e_{m+n-1})$.
If $m \neq 1$, then by Steps 2 and 3, we obtain

$$\begin{aligned}
0 &= [e_{m+n}e_{m+n}e_{m+n,i}]e_{m+n-1}\\
&= e_{m+n}\{e_{m+n}e_{m+n,i}e_{m+n-1}\} + (e_{m+n}e_{m+n-1})(e_{m+n}e_{m+n,i})\\
&= e_{m+n}\{e_{m+n}e_{m+n-1}e_{m+n,i}\} - b = a - 2b,
\end{aligned}$$

and

$$\begin{aligned}
0 &= ([e_{m+n}e_{m+n}e_{m+n-1}](e_{m+n}e_{m+n-1}))e_{m+n-2,i}\\
&= [e_{m+n}e_{m+n}e_{m+n-1}]e_{m+n,i} + ([e_{m+n}e_{m+n}e_{m+n-1}]e_{m+n-2,i})\\
&\qquad \times (e_{m+n}e_{m+n-1})\\
&= a + e_{m+n}\{e_{m+n}e_{m+n-1}e_{m+n-2,i}\}(e_{m+n}e_{m+n-1}) = a + b.
\end{aligned}$$

Hence we have $a = b = 0$. Similarly, if $m = 1$, then we get $a + b = 0$ and $a + 2b = 0$, which implies $a = b = 0$.

Step 5. For all $i, j \in \Omega$, we have $\{e_{m+n}e_{m+n,i}e_{m+n,j}\} = 0$.

If $j = m + n$ or $j = m + n - 1$, our assertion holds by Steps 2 and 4. We will use the downward induction on j. Suppose $j < m + n - 1$ and $\{e_{m+n}e_{m+n,i}e_{m+n,j}\} = 0$ for all $i \in \Omega$. Then we have

$$\begin{aligned} 0 &= \{e_{m+n}e_{m+n,i}e_{m+n,j}\}e_{j-1} \\ &= (e_{m+n}e_{m+n,i})e_{m+n,j} + \big((e_{m+n}e_{m+n,i})e_{j-1}\big)e_{m+n,j} \\ &= \{e_{m+n}e_{m+n,i}e_{m+n,j-1}\} + \delta_{ij}(e_{m+n}e_{m+n,i-1})e_{m+n,j} \\ &= \{e_{m+n}e_{m+n,i}e_{m+n,j-1}\}, \end{aligned}$$

which proves our claim.

Step 6. For all $i, j, k \in \Omega$, we have $[e_{m+n,i}e_{m+n,j}e_{m+n,k}] = 0$.

If $i = m + n$, Step 5 implies

$$e_{m+n}(e_{m+n,j}e_{m+n,k}) = (e_{m+n}e_{m+n,j})e_{m+n,k} + e_{m+n,j}(e_{m+n}e_{m+n,k}) = 0.$$

If $i < m + n$, by the above observation, we get

$$\begin{aligned} &[e_{m+n,i}e_{m+n,j}e_{m+n,k}] \\ &\qquad = (e_{m+n}e_{m+n-1,i})(e_{m+n,j}e_{m+n,k}) \\ &\qquad = e_{m+n}[e_{m+n-1,i}e_{m+n,j}e_{m+n,k}] \\ &\qquad\quad + (-1)^{d}[e_{m+n}e_{m+n,j}e_{m+n,k}]e_{m+n-1,i} \\ &\qquad = (-1)^{d'}\delta_{m+n,j}[e_{m+n}e_{m+n,i}e_{m+n,k}] \\ &\qquad\quad + (-1)^{d''}\delta_{m+n,k}[e_{m+n}e_{m+n,j}e_{m+n,i}] = 0, \end{aligned}$$

where $d, d', d'' \in \mathbb{Z}_2$.

It remains to prove the last relation. But it is an immediate consequence of Step 6. ∎

Let $X = E \cup H \cup F = \{e_i, h_i, f_i \mid i \in \Omega\}$ be a $\mathbb{Z}_2$-graded set, where $\Omega = \{1, 2, \ldots, m + n\}$ and $\tau = \{n\}$ is the set of odd index. Let R_+ be the set of relations in $E^{\#}$ given by:

I. e_ie_j $(m + n \geq i > j + 1 > 1)$,
II. $e_{ij}e_{i-1}$ $(m + n \geq i > j \geq 1)$,
III. $e_{ij}e_{i,j-1}$ $(m + n > i \geq j > 1)$,
IV. $e_{n+k,n-l}e_{n+k,n-l}$ $(m > k \geq 0,\ n > l \geq 0)$,

V. $[e_{m+n,i}e_{m+n,j}e_{m+n,j-1}]$ $(m+n \geq i \geq j > 1)$,

VI. $\{e_{m+n,i}e_{m+n,j}e_{m+n,i-1}\}$ $(m+n \geq i > j \geq 1)$,

VII. $(e_{m+n,i}e_{m+n,j})(e_{m+n,i}e_{m+n,j})$ $(m+n \geq i > n \geq j \geq 1)$,

VIII. $(e_{m+n,i}e_{m+n,j})(e_{m+n,i}e_{m+n,j-1})$ $(n \geq i > j > 1)$.

Let R_- be the set of relations in $F^{\#}$ obtained by replacing e_{ij}'s in R_+ by f_{ij}'s, and let $R(A,\tau) = R_+ \cup W \cup R_-$. Consider the Lie superalgebra $L = \mathscr{L}_X / \langle R(A,\tau)\rangle$. Then, by Lemma 4.3, there exists a surjective Lie superalgebra homomorphism ψ: $L \to \mathscr{G}$ defined by $e_i \mapsto e_i$, $h_i \mapsto h_i$, and $f_i \mapsto f_i$ $(i \in \Omega)$. Then we have:

THEOREM 4.4. *The set $R(A,\tau)$ of the relations in $\mathscr{L}_X$ is a Gröbner–Shirshov basis for the Lie superalgebra L.*

Proof. Set $R = R(A,\tau)$. As in the case of $sl(m,n)$, the set of $R(A,\tau)$-reduced super-Lyndon–Shirshov monomials is $B = B_+ \cup H \cup B_-$. We claim that the set of R_+-reduced Lyndon–Shirshov monomials in $\mathscr{L}_E$ is

$$B'_+ = \{e_{ij} \mid i \geq j\} \cup \{e_{m+n,i}e_{m+n,j} \mid i > j\}.$$

Let w be an R_+-reduced Lyndon–Shirshov monomial in $\mathscr{L}_E$. If $l(w) = 1$, there is nothing to prove. If $l(w) > 1$, then $w = uv$, where u, v are R_+-reduced Lyndon–Shirshov monomials. Hence by induction, we have $u, v \in B'_+$. We show that either $u = e_{m+n,j}$, $v = e_{m+n,l}$ with $j > l$ or $u = e_i$, $v = e_{i-1,l}$, which would prove our claim. We need to consider the following four cases.

Case 1. $u = e_{ij}$, $v = e_{kl}$ $(i \geq j,\ k \geq l)$:

Since uv is Lyndon–Shirshov, we have $(i,j) > (k,l)$ lexicographically. If $i = k = m+n$, then $u = e_{m+n,j}$, $v = e_{m+n,l}$ with $j > l$. If $i = k < m+n$, then $j - 1 \geq l$, and $\overline{e_{ij}e_{kl}}$ contains $\overline{e_{ij}e_{i,j-1}}$ as a subword. Hence w is not R_+-reduced by III. If $i = j > k$ and $k = i-1$, then u and v have the desired form and we are done. If $i = j > k$ and $k \leq i-2$, then w is not R_+-reduced by I. If $i > k$ and $i > j$, then we must have $k = i-1$, since $e_{ij} = e_i e_{i-1,j}$ and $e_{i-1,j} \leq e_{kl}$ by the definition of Lyndon–Shirshov monomials. Hence w is not R_+-reduced by II.

Case 2. $u = e_{kl}$, $v = e_{m+n,i}e_{m+n,j}$ $(i > j,\ k \geq l)$:

Since uv is Lyndon–Shirshov, we have $k = m+n$ and $l \geq i$. Then w is not R_+-reduced by V, since $\overline{w}$ contains $\overline{e_{m+n,i}(e_{m+n,i}e_{m+n,i-1})}$ as a subword.

Case 3. $u = e_{m+n,i}e_{m+n,j}$, $v = e_{kl}$ $(i > j,\ k \geq l)$:

Since uv is Lyndon–Shirshov, we have $e_{m+n,i} > e_{kl} \geq e_{m+n,j}$. It follows that $k = m+n$ and $i > l \geq j$. Hence $\overline{w}$ contains $\overline{\{e_{m+n,i}e_{m+n,j}e_{m+n,i-1}\}}$ as a subword, and w is not R_+-reduced by VI.

Case 4. $u = e_{m+n,i}e_{m+n,j}$, $v = e_{m+n,k}e_{m+n,l}$ $(i > j,\ k > l)$:

Since uv is Lyndon–Shirshov, we have $(i,j) > (k,l)$ and $e_{m+n,j} \leq e_{m+n,k}e_{m+n,l}$. Thus we have $j < k$ and either $i = k > j > l$ or $i > k > j$. If $i = k > j > l$, then $\overline{w}$ contains $\overline{(e_{m+n,i}e_{m+n,j})(e_{m+n,i}e_{m+n,j-1})}$ as a subword, and w is not R_+-reduced by VII or VIII. If $i > k > j$, then $\overline{w}$ contains $\overline{\{e_{m+n,i}e_{m+n,j}e_{m+n,i-1}\}}$ as a subword, and w is not R_+-reduced by VI.

Now, let w be an R_+-reduced super-Lyndon–Shirshov monomial in $\mathscr{L}_E$. Then w is Lyndon–Shirshov or $w = uu$ with u a Lyndon–Shirshov monomial in $E_1^{\#}$. If the latter is true, then we have the following three possibilities:

(i) $u = e_{n+k,n-l}$ $(m > k \geq 0,\ n > l \geq 0)$,

(ii) $u = e_{m+n,j}$ $(1 \leq j \leq n)$,

(iii) $u = e_{m+n,i}e_{m+n,j}$ $(m+n \geq i > n \geq j \geq 1)$.

But the cases (i) and (iii) cannot occur by IV and VII. Therefore the set of R_+-reduced super-Lyndon–Shirshov monomials is given by

$$B_+ = \{e_{ij} \mid i \geq j\} \cup \{e_{m+n,i}e_{m+n,j} | i > j\} \cup \{e_{m+n,j}e_{m+n,j} | 1 \leq j \leq n\}.$$

Similarly, we get

$$B_- = \{f_{ij} | i \geq j\} \cup \{f_{m+n,i}f_{m+n,j} | i > j\} \cup \{f_{m+n,j}f_{m+n,j} | 1 \leq j \leq n\}.$$

By Lemma 2.4 B spans L. Since ϕ and ψ are surjective, we have $\mathrm{card}(B) \geq \dim \mathrm{osp}(2m+1, 2n)$. But the number of elements of B is $2(m+n)^2 + m + 3n$, which is equal to the dimension of $\mathrm{osp}(2m+1, 2n)$. Hence B is a linear basis of L and by Proposition 2.9, $R = R(A,\tau)$ is a Gröbner–Shirshov basis for the Lie superalgebra L. ∎

Remark. The proof of Theorem 4.4 shows that the Lie superalgebras $L, \mathscr{G}(A,\tau)$ and $B(m,n) = \mathrm{osp}(2m+1, 2n)$ are all isomorphic. Hence Theorem 4.4 gives a Gröbner–Shirshov basis for the Lie superalgebra $B(m,n) = \mathrm{osp}(2m+1, 2n)$. Our argument also shows that $R(A,\tau)$ is actually a minimal Gröbner–Shirshov basis.

4.3. *The Lie Superalgebras of Type* $B(0,n)$ $(n > 0)$

Let E_{ij} denote the $(2n+1) \times (2n+1)$ matrix whose (i,j)-entry is 1 and all the other entries are 0. Set

$$\begin{aligned} x_i &= E_{i+1,i+2} - E_{n+i+2,n+i+1} \qquad (1 \leq i \leq n-1), \\ x_n &= \sqrt{2}\,(E_{1,2n+1} + E_{n+1,1}), \\ y_i &= E_{i+2,i+1} - E_{n+i+1,n+i+2} \qquad (1 \leq i \leq n-1), \\ y_n &= \sqrt{2}\,(E_{1n} - E_{2n+1,1}). \end{aligned} \tag{4.8}$$

Then the elements $x_i, y_i, z_i = [x_i, y_i]$ generate the Lie superalgebra $B(0, n) = \mathrm{osp}(1, 2n)$ $(n > 0)$ and x_n, y_n are the odd generators.

On the other hand, let $\Omega = \{1, 2, \dots, n\}$, $\tau = \{n\} \subset \Omega$, and consider the generalized Cartan matrix $A = (a_{ij})_{i,j \in \Omega}$ defined by

$$\begin{aligned} a_{n,n} &= 2, \qquad a_{n,n-1} = -2, \\ a_{ij} &= -1 \quad \text{if } |i-j| = 1, (i,j) \neq (n, n-1), \\ a_{ij} &= 0 \quad \text{if } |i-j| > 1. \end{aligned} \tag{4.9}$$

Let $\mathscr{G} = \mathscr{G}(A, \tau)$ be the Kac–Moody superalgebra associated with (A, τ) and denote by e_i, f_i, h_i $(i = 1, 2, \dots, n)$ the generators of $\mathscr{G}$. Then, by the same argument as in the proof of Lemma 4.3, we obtain:

LEMMA 4.5. *In the Kac–Moody superalgebra $\mathscr{G} = \mathscr{G}(A, \tau)$, we have*

$$\begin{gathered} e_{ij}e_{kl} = \delta_{j-1,k}e_{il} \quad \text{if } (i,j) \geq (k,l), n > k, \\ [e_{ni}e_{nj}e_{nk}] = 0 \qquad (i,j,k \in \Omega), \\ (e_{ni}e_{nj})(e_{nk}e_{nl}) = 0 \qquad (i,j,k,l \in \Omega). \end{gathered} \tag{4.10}$$

Let $X = E \cup H \cup F = \{e_i, h_i, f_i \mid i \in \Omega\}$ be a $\mathbb{Z}_2$-graded set, where $\Omega = \{1, 2, \dots, n\}$ and $\tau = \{n\} \subset \Omega$ is the set of odd index. Let R_+ be the set of relations in $E^\#$ given by:

I. e_ie_j $(n \geq i > j + 1 > 1)$,
II. $e_{ij}e_{i-1}$ $(n \geq i > j \geq 1)$,
III. $e_{ij}e_{i,j-1}$ $(n > i \geq j > 1)$,
IV. $[e_{n,i}e_{n,j}e_{n,j-1}]$ $(n \geq i \geq j > 1)$,
V. $\{e_{n,i}e_{n,j}e_{n,i-1}\}$ $(n \geq i > j \geq 1)$,
VI. $(e_{n,i}e_{n,j})(e_{n,i}e_{n,j-1})$ $(n \geq i > j > 1)$.

Let R_- be the set of relations in $F^\#$ obtained by replacing e_{ij}'s in R_+ by f_{ij}'s, and let $R(A, \tau) = R_+ \cup W \cup R_-$. Consider the Lie superalgebra $L = \mathscr{L}_X / \langle R(A, \tau) \rangle$. Then there is a surjective Lie superalgebra homomorphism ψ: $L \to \mathscr{G}$, and using the same argument as in the proof of Theorem 4.4, we obtain:

THEOREM 4.6. *The set $R(A, \tau)$ of the relations in $\mathscr{L}_X$ is a Gröbner–Shirshov basis for the Lie superalgebra L.*

Remark. The set of R_+-reduced super-Lyndon–Shirshov monomials in $\mathscr{L}_E$ is given by

$$B_+ = \{e_{ij} | i \geq j\} \cup \{e_{n,i}e_{n,j} | i \geq j\},$$

and the Lie superalgebras $L, \mathcal{G}(A,\tau)$, and $B(0,n) = \mathrm{osp}(1,2n)$ are all isomorphic. Moreover, $R(A,\tau)$ is a minimal Gröbner–Shirshov basis.

4.4. *The Lie Superalgebras of Type $C(n)$ $(n \geq 2)$*

Let E_{ij} denote the $(2n+1)\times(2n+1)$ matrix whose (i,j)-entry is 1 and all the other entries are 0. Set

$$\begin{aligned} x_1 &= E_{13} - E_{n+2,2}, \\ x_i &= E_{i+1,i+2} - E_{n+i+1,n+i} \qquad (2 \leq i \leq n-1), \\ x_n &= E_{n+1,2n}, \\ y_1 &= E_{31} + E_{2,n+2}, \\ y_i &= E_{i+2,i+1} - E_{n+i,n+i+1} \qquad (2 \leq i \leq n-1), \\ y_n &= E_{2n,n+1}. \end{aligned} \tag{4.11}$$

Then the elements $x_i, y_i, z_i = [x_i, y_i]$ $(i = 1,2,\ldots,n)$ are the generators of the Lie superalgebra $C(n) = \mathrm{osp}(2,2n-2)$, and x_1, y_1 are the odd generators.

Let $\Omega = \{1,2,\ldots,n\}$, $\tau = \{1\} \subset \Omega$ and consider the generalized Cartan matrix $A = (a_{ij})_{i,j\in\Omega}$ defined by

$$\begin{aligned} &a_{11} = 0, \qquad a_{12} = 1, \qquad a_{n-1,n} = -2, \\ &a_{ij} = -1 \quad \text{if } |i-j| = 1, (i,j) \neq (1,2), (n-1,n), \\ &a_{ij} = 0 \quad \text{if } |i-j| > 1. \end{aligned} \tag{4.12}$$

Let $\mathcal{G} = \mathcal{G}(A,\tau)$ be the Kac–Moody superalgebra associated with (A,τ) and denote by e_i, f_i, h_i $(i = 1,2,\ldots,n)$ the generators of $\mathcal{G}$. Then there is a surjective Lie superalgebra homomorphism $\phi\colon \mathcal{G} \to \mathrm{osp}(2,2n-2)$ given by $e_i \mapsto x_i$, $f_i \mapsto y_i$, $h_i \mapsto z_i$ $(i = 1,2,\ldots,n)$.

By a similar argument in the proof of Lemma 4.3, we can derive a more refined set of relations in $\mathcal{G}$, which gives a Gröbner–Shirshov basis for the Lie superalgebra $\mathrm{osp}(2,2n-2)$ $(n \geq 2)$. Since the argument is a variation of the one given in Lemma 4.3, we omit the proof here.

LEMMA 4.7. *In the Kac–Moody superalgebra $\mathcal{G} = \mathcal{G}(A,\tau)$, we have*

$$\begin{aligned} e_{ij}e_{kl} = \delta_{j-1,k}e_{il} \quad &\text{if } (i,j) \geq (k,l) \text{ and } k \neq n-1 \text{ when } i = n, \\ \{e_{ni}e_{n-1,j}e_{n-1,k}\} &= 0 \qquad (n > i), \\ \{e_{ni}e_{n-1,j}e_{nk}\} &= 0 \qquad (i,j,k \in \Omega), \\ (e_{ni}e_{n-1,j})(e_{n,k}e_{n-1,l}) &= 0 \qquad (i,j,k,l \in \Omega). \end{aligned} \tag{4.13}$$

Let $X = E \cup H \cup F = \{e_i, h_i, f_i \mid i \in \Omega\}$ be a $\mathbb{Z}_2$-graded set, where $\Omega = \{1, 2, \ldots, n\}$ and $\tau = \{1\} \subset \Omega$ is the set of odd index. Let R_+ be the set of relations in $E^{\#}$ given by:

I. $e_i e_j$ $(n \geq i > j+1 > 1)$,

II. $e_{ij} e_{i-1}$ $(n > i > j \geq 1)$,

III. $e_{ij} e_{i,j-1}$ $(n \geq i \geq j > 1)$,

IV. $e_{i1} e_{i1}$ $(n \geq i \geq 1)$,

V. $\{e_{n,i} e_{n-1,j} e_{n-1}\}$ $(n > j \geq i \geq 1)$,

VI. $\{e_{n,i} e_{n-1,j} e_{n,i-1}\}$ $(n > j \geq i > 1)$,

VII. $[e_{n,i} e_{n,i} e_{n-1}]$ $(n \geq i > 1)$,

VIII. $(e_{n,1} e_{n-1,j})(e_{n,1} e_{n-1,j})$ $(n > j > 1)$,

IX. $(e_{n,i} e_{n-1,j})(e_{n,i} e_{n-1,j-1})$ $(n > j > i > 1)$.

Let R_- be the set of relations in $F^{\#}$ obtained by replacing e_{ij}'s in R_+ by f_{ij}'s, and let $R(A, \tau) = R_+ \cup W \cup R_-$. Consider the Lie superalgebra $L = \mathscr{L}_X / \langle R(A, \tau) \rangle$. Then there is a surjective Lie superalgebra homomorphism ψ: $L \to \mathscr{G}$ defined by $e_i \mapsto e_i$, $f_i \mapsto f_i$, $h_i \mapsto h_i$ $(i \in \Omega)$. Moreover, we have:

THEOREM 4.8. *The set $R(A, \tau)$ of the relations in $\mathscr{L}_X$ is a Gröbner–Shirshov basis for the Lie superalgebra L.*

Proof. Since our argument is similar to the one for the proof of Theorem 4.4, we just give a sketch of the proof. We first prove that the set of $R(A, \tau)$-reduced Lyndon–Shirshov monomials in $\mathscr{L}_X$ is given by

$$B'_+ = \left\{ e_{ij} | i \geq j \right\} \cup \left\{ e_{n,i} e_{n-1,j} | n > j \geq i \geq 1 \text{ and } (i, j) \neq (1,1) \right\},$$

and conclude the set B_+ of R_+-reduced super-Lyndon–Shirshov monomials in $\mathscr{L}_E$ is equal to B'_+.

We see that $B = B_+ \cup H \cup B_-$ spans L, where

$$B_- = \left\{ f_{ij} | i \geq j \right\} \cup \left\{ f_{n,i} f_{n-1,j} | n > j \geq i \geq 1 \text{ and } (i, j) \neq (1,1) \right\}$$

is the set of R_--reduced super-Lyndon–Shirshov monomials in $\mathscr{L}_F$. The number of elements in B is $2n^2 + n - 2$, which is equal to the dimension of $\mathrm{osp}(2, 2n-2)$ $(n \geq 2)$. Hence the homomorphisms ϕ and ψ are isomorphisms, and B is a linear basis of L, which proves our assertion. ∎

Remark. The Lie superalgebras $L, \mathscr{G}(A, \tau)$, and $C(n) = \mathrm{osp}(2, 2n-2)$ are all isomorphic and $R(A, \tau)$ is a minimal Gröbner–Shirshov basis.

4.5. *The Lie Superalgebras of Type $D(m,n)$ $(m \geq 2,\ n > 0)$*

Let E_{ij} denote the $(2m+2n) \times (2m+2n)$ matrix whose (i,j)-entry is 1 and all the other entries are 0. Set

$$\begin{aligned}
x_i &= E_{2m+i,2m+i+1} - E_{2m+n+i+1,2m+n+i} \qquad (1 \leq i \leq n-1),\\
x_n &= E_{2m+n,1} + E_{m+1,2m+2n},\\
x_{n+i} &= E_{i,i+1} - E_{m+i+1,m+i} \qquad (1 \leq i \leq m-1),\\
x_{m+n} &= E_{m,2m-1} - E_{m-1,2m},\\
y_i &= E_{2m+i+1,2m+i} - E_{2m+n+i,2m+n+i+1} \qquad (1 \leq i \leq n-1),\\
y_n &= E_{1,2m+n} - E_{2m+2n,m+1},\\
y_{n+i} &= E_{i+1,i} - E_{m+i,m+i+1} \qquad (1 \leq i \leq m-1),\\
y_{m+n} &= E_{2m-1,m} - E_{2m,m-1}.
\end{aligned} \tag{4.14}$$

Then the elements $x_i, y_i, z_i = [x_i, y_i]$ $(i = 1, 2, \ldots, m+n)$ are the generators of the Lie superalgebra $D(m,n) = \mathrm{osp}(2m, 2n)$, and x_n, y_n are the odd generators.

Let $\Omega = \{1, 2, \ldots, m+n\}$, $\tau = \{n\}$, and consider the generalized Cartan matrix $A = (a_{ij})_{i,j \in \Omega}$ defined by

$$\begin{gathered}
a_{nn} = 0, \qquad a_{n,n+1} = 1, \qquad a_{m+n-2,m+n} = -1,\\
a_{m+n-1,m+n} = 0, \qquad a_{m+n,m+n-2} = -1, \qquad a_{m+n,m+n-1} = 0,\\
a_{ij} = -1 \quad \text{if } |i-j| = 1, \text{ and } (i,j) \neq (n, n+1),\\
(m+n-1, m+n), (m+n, m+n-1),\\
a_{ij} = 0 \quad \text{if } |i-j| > 1, \text{ and } (i,j) \neq (m+n-2, m+n),\\
(m+n, m+n-2).
\end{gathered} \tag{4.15}$$

Let $\mathscr{G} = \mathscr{G}(A, \tau)$ be the Kac–Moody superalgebra associated with (A, τ) and denote by e_i, f_i, h_i $(i = 1, 2, \ldots, m+n)$ the generators of $\mathscr{G}$. Then there is a surjective Lie superalgebra homomorphism $\phi\colon \mathscr{G} \to \mathrm{osp}(2m, 2n)$ given by $e_i \mapsto x_i$, $f_i \mapsto y_i$, $h_i \mapsto z_i$ $(i = 1, 2, \ldots, m+n)$.

We modify some of our notations:

(i) We neglect $e_{m+n,m+m-1}$; if $j \leq m+n-2$, we write $e_{m+n,j} = e_{m+n}e_{m+n-2,j}$.

(ii) We introduce a modified Kronecker's delta:

$$\hat{\delta}_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } i = j+1 = m+n-1,\\ 0 & \text{otherwise.} \end{cases}$$

In the following lemma, we list a set of relations in $\mathcal{G}$ which would yield a Gröbner–Shirshov basis for the Lie superalgebra $D(m, n) = \mathrm{osp}(2m, 2n)$ ($m \geq 2$, $n > 0$). We omit the proof which is similar to that of Lemma 4.3.

LEMMA 4.9. *In the Kac–Moody superalgebra $\mathcal{G} = \mathcal{G}(A, \tau)$, we have*

$$\begin{gathered}
e_{ij}e_{kl} = \hat{\delta}_{j-1,k}e_{il} \quad \text{if } (i,j) \geq (k,l),\ (i,k) \neq (m+n, m+n-1), \\
e_{m+n,i}e_{m+n-1,i} = 0 \quad \text{if } i > n, \\
e_{m+n,i}e_{m+n-1,i-1} = e_{m+n,i-1}e_{m+n-1,i} \quad \text{if } i \leq n, \\
\{e_{m+n,i}e_{m+n-1,j}e_{m+n-1,k}\} = 0 \qquad (i,j,k \in \Omega), \\
\{e_{m+n,i}e_{m+n-1,j}e_{m+n,k}\} = 0 \qquad (i,j,k \in \Omega), \\
(e_{m+n,i}e_{m+n-1,j})(e_{m+n,k}e_{m+n-1,l}) = 0 \qquad (i,j,k,l \in \Omega).
\end{gathered} \tag{4.16}$$

Let $X = E \cup H \cup F = \{e_i, h_i, f_i \mid i \in \Omega\}$ be a $\mathbb{Z}_2$-graded set, where $\Omega = \{1, 2, \ldots, m+n\}$ and $\tau = \{n\} \subset \Omega$ is the set of odd index. Let R_+ be the set of relations in $E^{\#}$ given by:

I. e_ie_j ($i > j+1$, $(i,j) \neq (m+n, m+n-2)$), $e_{m+n}e_{m+n-1}$,

II. $e_{ij}e_{i-1}$ ($m+n > i > j$), $e_{m+n,j}e_{m+n-2}$ ($m+n-2 \geq j$),

III. $e_{ij}e_{i,j-1}$ ($i \geq j > 1$) with $j \leq m+n-2$ when $i = m+n$, $e_{m+n}(e_{m+n}e_{m+n-2})$,

IV. $e_{m+n,i}e_{m+n-1,i}$ ($m+n-2 \geq i > n$), $e_{m+n,i}e_{m+n-1,i-1} - e_{m+n,i-1}e_{m+n-1,i}$ ($i \leq n$),

V. $e_{n+k,n-l}e_{n+k,n-l}$ ($m \geq k \geq 0$, $n > l \geq 0$),

VI. $\{e_{m+n,i}e_{m+n-1,j}e_{m+n-1}\}$ ($i < j < m+n$), $\{e_{m+n,i}e_{m+n-1,i}e_{m+n-1}\}$ ($i \leq n$),

VII. $\{e_{m+n,i}e_{m+n,i}e_{m+n-1}\}$ ($m+n-2 \geq i$),

VIII. $\{e_{m+n,i}e_{m+n-1,j}e_{m+n,i-1}\}$ ($1 < i < j < m+n$), $\{e_{m+n,i}e_{m+n-1,i}e_{m+n,i-1}\}$ ($i \leq n$),

IX. $(e_{m+n,i}e_{m+n-1,j})(e_{m+n,i}e_{m+n-1,j-1})$ ($n+1 < i+1 < j < m+n$), $(e_{m+n,i}e_{m+n-1,j})(e_{m+n,i}e_{m+n-1,j})$ ($i \leq n$, $i < j$).

Let R_- be the set of relations in $F^{\#}$ obtained by replacing e_{ij}'s by f_{ij}'s in R_+, and let $R(A, \tau) = R_+ \cup W \cup R_-$. Consider the Lie superalgebra $L = \mathcal{L}_X / \langle R(A, \tau) \rangle$. Then there is a surjective Lie superalgebra homomorphism ψ: $L \to \mathcal{G}$ defined by $e_i \mapsto e_i$, $f_i \mapsto f_i$, $h_i \mapsto h_i$ ($i \in \Omega$), and we have:

THEOREM 4.10. *The set $R(A, \tau)$ of the relations in $\mathcal{L}_X$ is a Gröbner–Shirshov basis for the Lie superalgebra L.*

Proof. As in the case of $C(n) = \mathrm{osp}(2, 2n - 2)$, we only give a brief sketch of the proof here. The set of $R_{\pm}$-reduced super-Lyndon–Shirshov monomials in $\mathscr{L}_E$ (resp., $\mathscr{L}_F$) is given by

$$B_+ = \{e_{ij} | i \geq j\} \cup \{e_{m+n,i} e_{m+n-1,j} | i < j \text{ or } i = j \leq n\},$$

$$B_- = \{f_{ij} | i \geq j\} \cup \{f_{m+n,i} f_{m+n-1,j} | i < j \text{ or } i = j \leq n\}.$$

Hence the number of elements in the set of $R(A, \tau)$-reduced super-Lyndon–Shirshov monomials in $\mathscr{L}_X$ is $2(m + n)^2 - m + n$, which is equal to the dimension of the Lie superalgebra $D(m, n) = \mathrm{osp}(2m, 2n)$. Therefore, B is a linear basis of L and $R(A, \tau)$ is a Gröbner–Shirshov basis for L. ∎

Remark. The Lie superalgebras $L, \mathscr{G}(A, \tau)$, and $D(m, n) = \mathrm{osp}(2m, 2n)$ are all isomorphic and $R(A, \tau)$ is a minimal Gröbner–Shirshov basis.

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